

# Retrofitting State Feedback Control of Networked Nonlinear Systems Based on Hierarchical Expansion

Tomonori Sadamoto<sup>1</sup>, Takayuki Ishizaki<sup>1</sup>, Jun-ichi Imura<sup>1</sup>, Henrik Sandberg<sup>2</sup>,  
and Karl Henrik Johansson<sup>2</sup>

**Abstract**—In this paper, we propose a systematic method to design *retrofit controllers* for networked nonlinear systems. The retrofit controller, which consists of a linear state feedback controller and a dynamical compensator, can improve the control performance for a subsystem of interest, while guaranteeing the stability of the whole closed-loop system. Towards the retrofit controller design, we introduce a type of state-space expansion, called *hierarchical expansion*. The cascade structure of the hierarchical expansion realization enables the systematic design of stabilizing controller for a low-dimensional linear model extracted from the subsystem of interest. As a result, we can design a retrofit controller without explicit consideration of the dynamics of subsystems other than the subsystem of interest. The effectiveness of the proposed method is demonstrated through a power network example.

## I. INTRODUCTION

Many infrastructure and industrial processes, e.g., power networks [1], [2], transportation networks [3] and manufacturing plants [4], are constructed by the integration of physical processes and various controllers having different objectives. For example, in the power system control, several distributed controllers, called primary controllers, are installed into individual generators to maintain the balance between the amounts of generation and load. In addition to the primary controllers, secondary controllers are also used to adjust the reference signals for primary controllers. Even though the controllers are individually designed according to their different objectives, the overall power network practically works well. As we can see from this example, it is desirable to systematically retrofit additional controllers to accomplish individual objectives in a distributed fashion.

Although various distributed controller design methods have been proposed in, e.g., [5], [6], the resultant control systems do not generally have practical retrofitability. This is due to the fact that the distributed controllers are designed in a centralized fashion. As being concerned with this matter, the notion of distributed design has been introduced in [7], which discusses a performance limitation of controllers designed in a distributed manner. This work has been extended in [8] to networked control systems having the input ports whose number is the same as the dimension of the state variables. In addition, a distributed design method in terms

of the  $\mathcal{L}_1$ -induced norm has been developed for positive systems [9]. However, it is not straightforward to generalize these methods to a broader class of linear systems because they focus on particular class of linear systems. Moreover, from a viewpoint of applications, practical network systems inevitably involve some nonlinearities [10]. In view of this, towards realizing more practical control systems, it is crucial to devise a method to design controllers that can be retrofitted as being complying with realistic requirements for application.

Against this background, this paper proposes a systematic method to design *retrofit controllers* for networked nonlinear systems that can improve some degree of control performance for a subsystem of interest. In this paper, a control strategy satisfying the following two requirements is called *retrofitting control*: retrofitting additional controllers does not spoil the stability of the original closed-loop system, and the retrofit controller design is performed without explicit use of the model information of subsystems other than the subsystem of interest.

Towards systematic retrofit controller design, we introduce a type of state-space expansion called *hierarchical expansion*. The feature of the hierarchical expansion realization is its cascade structure, in which a low-dimensional linear model extracted from the subsystem of interest is placed in the upstream part and the preexisting nonlinear dynamics is placed in the downstream part. The low-dimensional linear model can be obtained by model reduction techniques [11]. By virtue of the cascade structure, the stabilization of each of upstream and downstream parts can be done independently. The proposed method is useful for large-scale network systems such as power grids composed of a number of areas. For example, the retrofit controller for an area, which is designed without explicit consideration of the dynamics of other areas, enables to improve frequency convergence rate in the area of interest. The retrofitting can be performed without spoiling the stability of the whole power grid. This will be demonstrated numerically in Section V.

As a preliminary work, the authors in [12] have introduced a type of state-space expansion for linear network systems, where the expanded state space represents the dynamics of disjoint subsystems. In [13], by interpreting a set of controllers, stabilizing disjoint subsystems, as dynamical systems installed into the original network systems, the authors dealt with the stabilization of a class of evolving network systems. In [14], a similar type of state-space expansion is used for network system control on multiple time

<sup>1</sup>Department of Mechanical and Environmental Informatics, Graduate School of Information Science and Engineering, Tokyo Institute of Technology; 2-12-1, Meguro, Tokyo, Japan:  
{sadamoto@cyb., ishizaki@, imura@}mei.titech.ac.jp

<sup>2</sup>School of Electrical Engineering, Automatic Control, Royal Institute of Technology (KTH), SE-100 44 Stockholm, Sweden:  
{hsan,kallej}@ee.kth.se

scales in a discrete-time setting. Compared to the state-space expansion in [12], [13], [14], the framework of hierarchical expansion introduced in this paper is more general, because the hierarchical expansion can handle a class of nonlinear systems.

This paper is organized as follows: In Section II, we describe a design problem of retrofit controllers for nonlinear networked systems. To solve it in a systematic manner, we introduce the hierarchical expansion in Section III. In Section IV-A, we show that the retrofit controller has the potential to improve a degree of control performance for a subsystem of interest, without spoiling the stability of the preexisting networked nonlinear system. In Section IV-B, we provide a design procedure for constructing low-dimensional retrofit controllers. In Section V, the effectiveness of the proposed retrofitting control is shown through a power network example. Finally, concluding remarks are provided in Section VI.

*Notation:* We denote the identity matrix by  $I$ , the image of a matrix  $M$  by  $\text{im } M$ , the pseudoinverse of a full-column rank matrix  $P$  by  $P^\dagger$ , which is not necessarily the Moore-Penrose pseudoinverse, the negative (positive) semidefinite symmetric matrix  $M$  by  $M \preceq 0$  ( $M \succeq 0$ ), the  $\mathcal{L}_2$ -norm of a square-integrable function  $f(\cdot)$  by

$$\|f(t)\|_{\mathcal{L}_2} := \sqrt{\int_0^\infty \|f(t)\|^2 dt}.$$

With  $\mathbb{N} = \{1, \dots, N\}$ , we denote the block-diagonal matrix having matrices  $M_i$  for  $i \in \mathbb{N}$  on its diagonal blocks by

$$\text{diag}(M_i)_{i \in \mathbb{N}} = \text{diag}(M_1, \dots, M_N),$$

where we omit the subscript of  $i \in \mathbb{N}$  if there is no chance of confusion. The  $\mathcal{H}_\infty$ -norm of a stable proper transfer matrix  $G$  is defined by

$$\|G(s)\|_{\mathcal{H}_\infty} := \sup_{\omega \in \mathbb{R}} \|G(j\omega)\|.$$

Throughout this paper, a dynamical system

$$\Sigma : \dot{x} = f(x, u)$$

is said to be stable in the sense that  $\Sigma$  is globally input-to-state stable [10]. By the abuse of the terminology, the autonomous system  $\dot{x} = f(x)$  is said to be stable in the sense that the system is globally asymptotically stable. A map  $\mathcal{F}(\cdot)$  is said to be a dynamical map if the triplet  $(x, u, y)$  with  $y = \mathcal{F}(u)$  solves a system of differential equations

$$\dot{x} = f(x, u), \quad y = g(x, u)$$

with some functions  $f(\cdot, \cdot)$  and  $g(\cdot, \cdot)$ , and an initial value  $x(0)$ . Unless otherwise stated, the initial value is supposed to be zero for dynamical maps.

## II. PROBLEM FORMULATION

Let us consider a large-scale networked nonlinear system denoted by  $\mathcal{S}$ . Examples of  $\mathcal{S}$  include the IEEE 39 bus system in [15], whose interconnection structure is shown in Fig. 1. We assume that  $\mathcal{S}$  is stabilized by preexistent controllers such as primary controllers. For this networked system, we

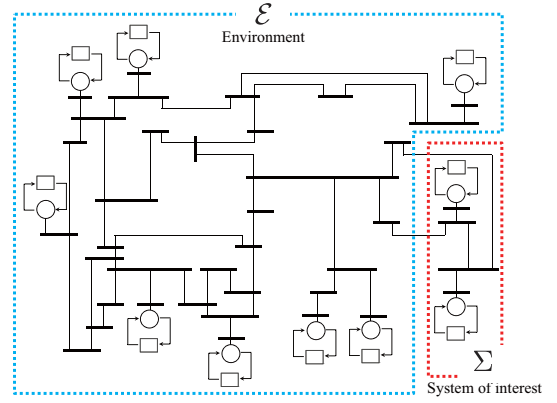


Fig. 1. Example of  $\mathcal{S}$ : IEEE 39 bus system

consider adding a controller to a local area of interest that has the potential to improve a degree of control performance, e.g., frequency convergence rate, for the corresponding area. In this paper, let  $\Sigma$  denote the specific local area (subsystem) whose control performance is to be improved by retrofitting a local controller, whereas let  $\mathcal{E}$  denote the set of subsystems other than  $\Sigma$ , which we call *environment*; see Fig. 1. Note that  $\mathcal{E}$  may be larger in scale, i.e., it may have higher dimension than that of  $\Sigma$ .

We denote the state of  $\Sigma$  by  $x$  and that of  $\mathcal{E}$  by  $\xi$ . The dynamics of  $\mathcal{E}$  is described as

$$\mathcal{E} : \begin{cases} \dot{\xi} = g(\xi, x) \\ w = h(\xi, x) \end{cases} \quad (1)$$

where  $g(\cdot, \cdot)$  and  $h(\cdot, \cdot)$  are static functions, which are possibly nonlinear, and  $w$  is an interconnection signal injected to  $\Sigma$ . To simplify the notation, we describe (1) as

$$w = \mathcal{E}(x),$$

where  $\mathcal{E}(\cdot)$  denotes the corresponding dynamical map. We give the dynamics of  $\mathcal{S}$  as

$$\mathcal{S} : \begin{cases} \dot{x} = Ax + f(x) + h(\xi, x) + Bu \\ \dot{\xi} = g(\xi, x) \end{cases} \quad (2)$$

where  $f(\cdot)$  is a static function representing nonlinearity in the vector field, and  $u$  is a control input. Note that the first equation in (2) represents the dynamics of  $\Sigma$ . To simplify the notation, we describe (2) as

$$\mathcal{S} : \dot{x} = Ax + f(x) + \mathcal{E}(x) + Bu. \quad (3)$$

In the rest of this paper, we assume that the state  $x$  and the interconnection signal  $w$  are both measurable. In what follows, we denote the dimension of  $x$  by  $n$ .

For this networked system, let us consider designing a *retrofit controller* to improve a degree of control performance of  $\Sigma$ , such as  $\|x\|_{\mathcal{L}_2}$ . The requirements of the retrofitting control are twofold: the retrofit of additional controllers does not spoil the stability of the closed-loop system, and the retrofit controller design is performed without explicit use of the model information of  $\mathcal{E}$ . The second requirement would

be indispensable because the modeling of the entire network is not necessarily easy for large-scale systems. Furthermore, it would be desirable to provide a clear guideline of controller tuning.

One simple approach for retrofitting control is to design a linear state feedback controller for a linear model of  $\Sigma$ . To derive a linear model to be controlled, we can utilize model reduction techniques [11]. More specifically, by denoting the linear part of  $\mathcal{S}$  as

$$\dot{x} = Ax + Bu \quad (4)$$

a model reduction technique provides a lower-dimensional approximant such that

$$x \approx P\hat{x}, \quad \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u, \quad (5)$$

where  $P \in \mathbb{R}^{n \times \hat{n}}$  is a full-column rank matrix and  $\hat{A}$  and  $\hat{B}$  are matrices having dimensions compatible with  $P$ . As long as the approximation is fine, we can expect that the state feedback control

$$u = \hat{F}P^\dagger x \quad (6)$$

stabilizes (4) if  $\hat{F}$  is designed such that  $\hat{A} + \hat{B}\hat{F}$  is Hurwitz.

However, in general, the control (6) may spoil the stability of (3), ensured by the interconnection with the environment. This is due to the fact that the nonlinearity  $f(x)$  and  $\mathcal{E}(x)$  neglected in (3) are stimulated by the feedback control of (6). To prevent the stimulation to the nonlinearity, we consider adding a compensation signal to (6) in a way that

$$u = \hat{F}P^\dagger x - \hat{\mathcal{F}}(x, w) \quad (7)$$

where the dynamical map  $\hat{\mathcal{F}}(\cdot, \cdot)$  is an  $\hat{n}$ -dimensional dynamical compensator to be designed. Without loss of generality, we assume that the dimension  $\hat{n}$  is less than or equal to the dimension  $n$  of  $x$ . In this formulation, we do not assume the stabilizability of the pair  $(A, B)$ , but we consider extracting a stabilizable linear model associated with  $(\hat{A}, \hat{B})$  by virtue of model reduction.

In summary, we aim at designing a retrofit controller such that

- 1)  $\hat{F}$  and  $P$  to control a linear model associated with the pair  $(\hat{A}, \hat{B})$ , and
- 2)  $\hat{n}$ -dimensional  $\hat{\mathcal{F}}(\cdot, \cdot)$  to prevent the stimulation to the nonlinear dynamics in (3).

This retrofit controller is implemented to improve an index of control performance of  $\Sigma$ , e.g., frequency convergence rate in the local area represented as  $\Sigma$  in Fig. 1. To solve this design problem in a tractable manner, in the following section, we introduce a fundamental mathematical tool with regard to state-space expansion, which we call *hierarchical expansion*.

### III. HIERARCHICAL EXPANSION

For  $\mathcal{S}$  in (3), let  $\mathcal{H}(\cdot) := f(\cdot) + \mathcal{E}(\cdot)$ . Under the same notation as that in the previous section, let us consider the state-space expansion of  $\mathcal{S}$  described as

$$\begin{cases} \dot{\phi} = A\phi + \mathcal{H}(\phi + P\hat{\phi}) + (AP - P\hat{A})\hat{\phi} \\ \dot{\hat{\phi}} = \hat{A}\hat{\phi} + \hat{B}u, \end{cases} \quad (8)$$

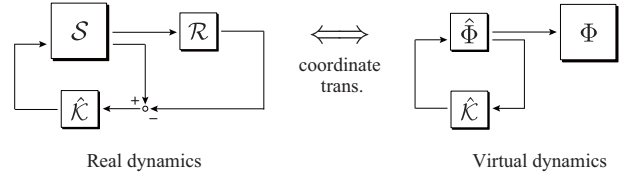


Fig. 2. Signal-flow diagrams of virtual and real dynamics with a controller  $\hat{\mathcal{K}}(\cdot)$ , where  $\hat{\Phi}$  and  $\Phi$  denote the dynamics of  $\hat{\phi}$  and  $\phi$  in (8).

where  $\bar{P}$  satisfies  $PP^\dagger + \bar{P}\bar{P}^\dagger = I$ , and  $\phi \in \mathbb{R}^n$  and  $\hat{\phi} \in \mathbb{R}^{\hat{n}}$ . Note that (8) has a cascade structure in which a linear system described by the pair  $(\hat{A}, \hat{B})$ , which is a linear model of (3), is placed in the upstream part. To see a relation between (8) and (3), let us consider the coordinate transformation of

$$x = \phi + P\hat{\phi}, \quad \hat{x} = P^\dagger \phi, \quad (9)$$

whose inverse is given by

$$\phi = \bar{P}\bar{P}^\dagger x + P\hat{x}, \quad \hat{\phi} = P^\dagger x - \hat{x}. \quad (10)$$

Then, we have

$$\begin{cases} \dot{\hat{x}} = Ax + \mathcal{H}(x) + P\hat{B}u \\ \dot{\hat{x}} = \hat{A}\hat{x} + P^\dagger \mathcal{H}(x) + (P^\dagger A - \hat{A}P)x, \end{cases} \quad (11)$$

which has another cascade structure. Note that this redundant realization involves the dynamics of (3) in the upstream part if  $P\hat{B} = B$  holds. In this paper, we refer to this state-space expansion of (3) as *hierarchical expansion*.

We can say from the derivation above that “the virtual realization in (8) is to be observed as the redundant realization (11) in the real world.” The relation between the real and virtual realizations is given as the coordinate transformation in (9). In this formulation, we provide the following result:

*Lemma 1:* With the notation above, consider the cascade interconnection of systems

$$\begin{aligned} \mathcal{S} : & \begin{cases} \dot{x} = Ax + \mathcal{H}(x) + Bu \\ y = Cx \end{cases} \\ \mathcal{R} : & \begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + P^\dagger \mathcal{H}(x) + (P^\dagger A - \hat{A}P)x \\ \hat{y} = \hat{C}\hat{x}. \end{cases} \end{aligned} \quad (12)$$

Assume that  $\mathcal{S}$  is stable and

$$P\hat{B} = B, \quad \hat{C}P^\dagger = C. \quad (13)$$

Then the feedback system of  $\mathcal{S}$  and  $\mathcal{R}$  interconnected by

$$u = \hat{\mathcal{K}}(y - \hat{y}) \quad (14)$$

is stable for any dynamical map  $\hat{\mathcal{K}}(\cdot)$  if and only if the feedback system

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{\mathcal{K}}(\hat{C}\hat{x}) \quad (15)$$

is stable.

*Proof:* First, we show the sufficiency. Suppose that (15) is stable. By implementing the controller of

$$u = \hat{\mathcal{K}}(\hat{C}\hat{\phi}) \quad (16)$$

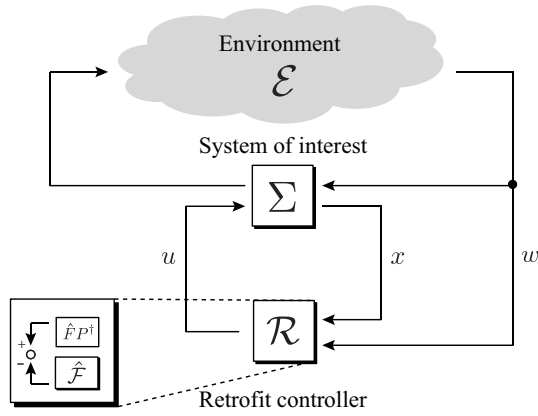


Fig. 3. Whole control system

to the virtual realization in (8), we have

$$\begin{cases} \dot{\hat{\phi}} = A\hat{\phi} + \mathcal{H}(\hat{\phi} + P\hat{\phi}) + (AP - P\hat{A})\hat{\phi} \\ \dot{\hat{\phi}} = \hat{A}\hat{\phi} + \hat{B}\hat{\mathcal{K}}(\hat{C}\hat{\phi}). \end{cases} \quad (17)$$

Since the dynamics with respect to  $\hat{\phi}$  in (17) is stable, we have

$$\lim_{t \rightarrow \infty} \hat{\phi}(t) = 0.$$

Hence, it follows from (16) that

$$\lim_{t \rightarrow \infty} u(t) = 0.$$

In addition, since  $\mathcal{S}$  is globally input-to-state stable with respect to  $u$ ,  $\lim_{t \rightarrow \infty} x(t) = 0$  follows. Then, the second equation in (10) shows  $\lim_{t \rightarrow \infty} \hat{x}(t) = 0$ . Thus, the whole system feedback by  $u$  in (14) is globally asymptotically stable. Next, we show the necessity by contraposition. Namely, supposing that (15) is unstable for some  $\hat{\mathcal{K}}(\cdot)$ , we show that the feedback system of  $\mathcal{S}$  and  $\mathcal{R}$  interconnected by (14) is unstable. It follows from (17) that  $\lim_{t \rightarrow \infty} \|\hat{\phi}(t)\| \rightarrow \infty$ . Thus, it follows from (16) that  $\lim_{t \rightarrow \infty} \|u(t)\| \rightarrow \infty$ . Hence, from (12), we have  $\lim_{t \rightarrow \infty} \|x(t)\| \rightarrow \infty$ . This completes the proof. ■

For Lemma 1, it should be emphasized that the whole control system is stable as long as  $\hat{\mathcal{K}}(\cdot)$  stabilizes a linear model described by  $(\hat{A}, \hat{B})$ , which is extracted from  $\mathcal{S}$  in (12). In other words,  $\hat{\mathcal{K}}(\cdot)$  can be designed without explicit consideration of the nonlinear dynamics  $\mathcal{H}(\cdot)$ .

The relation between the virtual and real dynamics with  $\hat{\mathcal{K}}(\cdot)$  is depicted in Fig. 2, where the dynamics of  $\hat{\phi}$  and  $\phi$  in (8) are denoted by  $\hat{\Phi}$  and  $\Phi$ , respectively. From this figure, we see that  $\mathcal{R}$  in (12) can be regarded as a type of observers that provides a compensation signal to the controller  $\hat{\mathcal{K}}(\cdot)$ .

#### IV. RETROFITTING CONTROL

##### A. Retrofit Controller Design

Specializing Lemma 1 to the retrofit controller design problem described in Section II, we have the following theorem:

*Theorem 1:* Given  $\mathcal{S}$  in (3), consider

$$\mathcal{R} : \begin{cases} \dot{\hat{x}} = P^\dagger AP\hat{x} + P^\dagger f(x) + P^\dagger w + P^\dagger A\overline{P}P^\dagger x \\ u = \hat{F}P^\dagger x - \hat{F}\hat{x} \end{cases} \quad (18)$$

with  $P$  such that

$$\text{im } B \subseteq \text{im } P. \quad (19)$$

Then, the interconnected system composed of  $\mathcal{S}$  and  $\mathcal{R}$  is stable for any  $\hat{F}$  such that  $P^\dagger AP + P^\dagger B\hat{F}$  is Hurwitz.

*Proof:* Let

$$\hat{A} = P^\dagger AP, \quad \hat{B} = P^\dagger B$$

in (12). Then  $P^\dagger A - \hat{A}P = P^\dagger A\overline{P}P^\dagger$  and the first assumption in (13) holds if (19) holds. Furthermore, taking

$$C = P^\dagger, \quad \hat{\mathcal{K}}(\hat{\phi}) = \hat{F}\hat{\phi}, \quad \mathcal{H}(\cdot) = f(\cdot) + \mathcal{E}(\cdot),$$

the feedback control of (14) turns out to be (18). ■

For Theorem 1, it should be emphasized that the retrofit controller  $\mathcal{R}$  in (18) can be designed without explicit consideration of the model information of  $\mathcal{E}$ . Theorem 1 shows that the retrofit controller  $\mathcal{R}$  theoretically guarantees the stability of the closed-loop system as long as  $\hat{F}$  makes  $P^\dagger AP + P^\dagger B\hat{F}$  Hurwitz. The structure of the closed-loop system is shown in Fig. 3. Clearly, the dynamical map from  $x$  and  $w$  to  $\hat{F}\hat{x}$  in (18) corresponds to the dynamical map  $\hat{\mathcal{F}}(\cdot, \cdot)$  in (7).

By virtue of the cascade structure in (17), not only the stability analysis as in Theorem 1 but also a control performance analysis can be done. For simplicity, let us consider the initial value response in the case where the system associated with the downstream part of (17), namely

$$\dot{\phi} = A\phi + \mathcal{H}(\phi + \delta) + \overline{P}P^\dagger A\delta$$

satisfies that there exist class- $\mathcal{K}$  functions  $\beta(\cdot)$  and  $\gamma(\cdot)$  satisfying

$$\|\phi\|_{\mathcal{L}_2} \leq \beta(\|\phi(0)\|) + \gamma(\|\delta\|_{\mathcal{L}_2})$$

for any  $\mathcal{L}_2$ -bounded signal  $\delta$ ; see [10] for the definition of class- $\mathcal{K}$  functions. Let

$$\delta := P e^{(P^\dagger AP + P^\dagger B\hat{F})t} \hat{\phi}(0). \quad (20)$$

In this formulation, it follows from (9) and the triangle inequality of the  $\mathcal{L}_2$ -norm that

$$\|x\|_{\mathcal{L}_2} \leq \beta(\|\phi(0)\|) + \gamma'(\|\delta\|_{\mathcal{L}_2}), \quad \gamma'(\cdot) := 1 + \gamma(\cdot). \quad (21)$$

Note that  $\gamma'(\cdot)$  is independent of  $\hat{F}$ . Thus, if  $\hat{F}$  is tuned such that  $\|\delta\|_{\mathcal{L}_2}$  is made smaller, then the entire convergence rate can also be made smaller in the sense of the bound in (21). Furthermore, if

$$\text{im } x(0) \subseteq \text{im } P, \quad (22)$$

we can take  $\phi(0) = 0$  because there exists  $\hat{\phi}(0)$  such that  $x(0) = P\hat{\phi}(0)$ . Then,  $\beta(\|\phi(0)\|) = 0$ , which yields a tighter bound of  $\|x\|_{\mathcal{L}_2}$ .

*Remark 1:* As a control method aiming at controlling a linear dynamics associated with the original nonlinear system, the control method based on feedback linearization has been proposed in the literature, e.g., [16]. In this method, a nonlinear compensator is designed such that the nonlinearity of the original system is exactly included in an unobservable subspace. On the other hand, in the proposed method, the nonlinear dynamics in (17) has an influence on the output function  $x = \phi + P\hat{\phi}$ . In other words, the nonlinear dynamics  $\mathcal{H}(\cdot)$  is not unobservable in general. From this viewpoint, the proposed control method clearly differs from the control method based on feedback linearization.

*Remark 2:* Generalization to the case of dynamical output feedback controllers can be done by replacing  $u$  in (7) and (18) with

$$u = \hat{\mathcal{K}}(y - \hat{y}), \quad y = Cx, \quad \hat{y} = CP\hat{x}, \quad (23)$$

where  $\hat{\mathcal{K}}(\cdot)$  is a dynamical map stabilizing the system

$$\dot{\hat{x}} = P^\dagger AP\hat{x} + P^\dagger B\hat{\mathcal{K}}(\hat{y}).$$

However, this generalization does not make sense in practice, because implementation of  $\hat{\mathcal{F}}(\cdot, \cdot)$  in (18) requires another output signal  $\gamma = \bar{P}^\dagger x$ . Note that, if we take the output signal  $y$  in (23) as a map of  $\gamma$ , i.e.  $y = M\gamma$  with a matrix  $M$ , then

$$CP = M\bar{P}^\dagger P = 0$$

which follows from the fact that  $\bar{P}$  is complementary to  $P$ . Thus no stabilizing controller  $\hat{\mathcal{K}}(\cdot)$  in (23) is obtained unless we measure  $y$  different from  $\gamma$ . As we see here, the output signals  $y$  and  $\gamma$  should be complementary to each other. Indeed, the combination of  $y$  and  $\gamma$  used in the control of (7) corresponds to the full information of  $x$ . In this sense, the retrofit controller is, in principle, implemented as a dynamical controller that uses the full state information.

*Remark 3:* We have assumed that the feedback signal from the environment  $w = \mathcal{E}(\cdot)$  is measurable. However,  $w$  is not always measurable, e.g., it is difficult to measure all temperatures of nodes connecting to environment in a spatially discretized thermal diffusion networks. To relax the assumption of the availability of  $w$ , we consider constructing  $P$  as follows:

Let a full-column rank matrix  $E$  denote an input port of  $w$  to  $x$ , i.e.,  $E$  satisfies

$$\mathcal{E}(\cdot) = E\mathcal{E}'(\cdot)$$

for a dynamical map  $\mathcal{E}'(\cdot)$ . In this case, we find  $P$  satisfying

$$\text{im } E \subseteq \text{im } \bar{P}. \quad (24)$$

Note that we can always find a matrix  $P$  such that the condition (24) holds as long as the column vectors of  $B$  and  $E$  are linearly independent. In this situation, we can reduce the assumption of the availability of  $w$  from environment because  $P^\dagger w = 0$ . Thus we can implement  $\mathcal{R}$  in (18) as a retrofit controller that relies only on the state feedback of  $\hat{F}P^\dagger x$ .

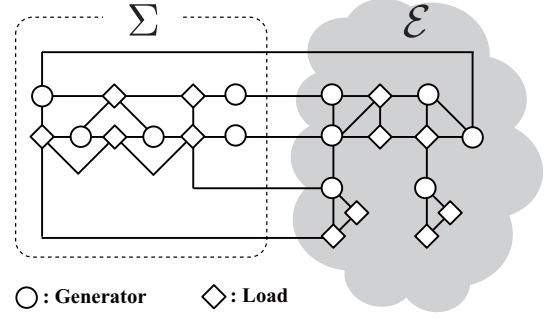


Fig. 4. Power network model

### B. Low-dimensional Retrofit Controller Design Procedure

We show a procedure to design a retrofit controller  $\mathcal{R}$  in (18). One policy for designing  $\hat{F}$  and  $P$  is to make  $\|\delta\|_{\mathcal{L}_2}$  small where  $\delta$  is defined as (20). If once we find  $P$  such that (5) and (19) hold, we can systematically find a suitable gain  $\hat{F}$  by existing state feedback design methods. This leads to  $\mathcal{R}$  by (18). Thus, in the following, we only consider how to find a suitable matrix  $P$ .

For simplicity, we assume that  $A$  is Hurwitz. A similar argument is valid also for unstable systems. Denoting  $G(s) := (sI - A)^{-1}B$ , we consider finding  $P$  such that (19) holds,  $\hat{G}_P(s) := P(sI - P^\dagger AP)^{-1}P^\dagger B$  is stable, and

$$\|G(s) - \hat{G}_P(s)\|_{\mathcal{H}_\infty} \quad (25)$$

is sufficiently small. This can be done by the balanced truncation with a slight modification, explained as follows. Let a nonsingular matrix  $V$  be given such that the system described by the triple  $(VAV^{-1}, VB, V^{-1})$  is a balanced realization of the system described by the triple  $(A, B, I)$ . Then, with  $E_k \in \mathbb{R}^{n \times k}$ , whose columns are eigenvectors associated with the first to  $k$ th largest eigenvalues of  $X$ , the standard balanced truncation corresponds to the case where we give  $P = V^{-1}E_k$  and  $P^\dagger$  being the Moore-Penrose pseudoinverse of  $P$ .

As a minor modification of this, to satisfy (19), we construct  $P$  by adding the bases of  $B$  to  $V^{-1}E_k$ . More specifically, parametrizing

$$P = V^{-1}W,$$

we find  $W$  such that

$$\text{im } W = \text{im } E_k + \text{im } VB.$$

Then, the constructed  $\hat{G}_P(s)$  is stable and satisfies (19) and the approximation error in (25) is expected to be small if  $k$  is large.

## V. NUMERICAL EXAMPLE

### A. Power Networks

We deal with a power network model [17] shown in Fig. 4. The power network is composed of 11 generators and 12 loads. We denote the index sets of generators and loads by

$$\mathbb{I}^G := \{1, \dots, 11\}, \quad \mathbb{I}^L := \{11, \dots, 23\}.$$

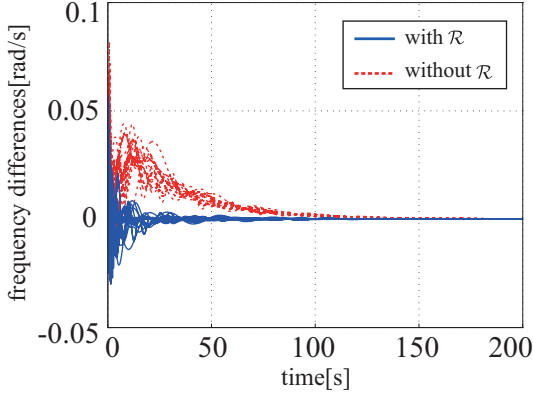


Fig. 5. Frequency differences of all generators and loads

In addition, the  $i$ th node represents the  $i$ th generator or the  $i$ th load. Furthermore, the index set of neighboring nodes connecting to the  $i$ th node is denoted by  $\mathcal{N}_{i,j}$ .

The dynamics of each generator is described by

$$\begin{cases} \dot{\zeta}_i = A_i \zeta_i + b_i u_i + r_i \sum_{j \in \mathcal{N}_{i,j}} Y_{i,j} \sin(\delta_j - \delta_i) \\ \dot{\delta}_i = c_i \zeta_i \end{cases} \quad (26)$$

where  $Y_{i,j}$  is the admittance between the  $i$ th and  $j$ th nodes scaled by their voltage amplitude, and each state variable of  $\zeta_i \in \mathbb{R}^4$  denotes the phase angle difference, angular velocity difference, mechanical input difference, and valve position difference. In addition,  $u_i \in \mathbb{R}$  is the angular velocity difference command, and  $\delta_i$  is the phase angle difference. Furthermore, the system matrices in (26) are given by

$$A_i := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -D_i/M_i & -1/M_i & 0 \\ 0 & 0 & -1/T_i & 1/T_i \\ 0 & 1/K_i & 0 & -R_i/K_i \end{bmatrix} \quad (27)$$

$$b_i := \frac{1}{K_i} e_4^4, \quad c_i := (e_1^4)^\top, \quad r_i := \frac{1}{M_i} e_2^4$$

where  $e_i^n \in \mathbb{R}^n$  is the  $i$ th column of  $I_n$  and  $M_i$ ,  $D_i$ ,  $T_i$ ,  $K_i$  and  $R_i$  denote the inertia constant, damping coefficient, turbine time constant, governor time constant, and droop characteristic, respectively. These parameters are randomly chosen from the ranges of  $[0.1, 1]$ ,  $[0.1, 1.5]$ ,  $[0.01, 10]$ ,  $[0.1, 2]$  and  $[0.1, 2]$ , respectively. Note that the unit of all physical variables is [p.u.].

Next, the dynamics of each load is described by

$$\begin{cases} \dot{\zeta}_i = A_i \zeta_i + r_i \sum_{j \in \mathcal{N}_{i,j}} Y_{i,j} \sin(\delta_j - \delta_i) \\ \dot{\delta}_i = c_i \zeta_i \end{cases} \quad (28)$$

where each state variable of  $\zeta_i \in \mathbb{R}^2$  denotes the phase angle difference and angular velocity difference, and  $\delta_i \in \mathbb{R}$  denotes the phase angle difference. Furthermore, the system matrices in (28) are given by

$$A_i := \begin{bmatrix} 0 & 1 \\ 0 & -D_i/M_i \end{bmatrix}, \quad r_i := \frac{1}{M_i} e_2^2, \quad c_i := (e_1^2)^\top$$

where  $M_i$  and  $D_i$  denote the inertia constant and damping coefficient, respectively. These parameters are randomly cho-

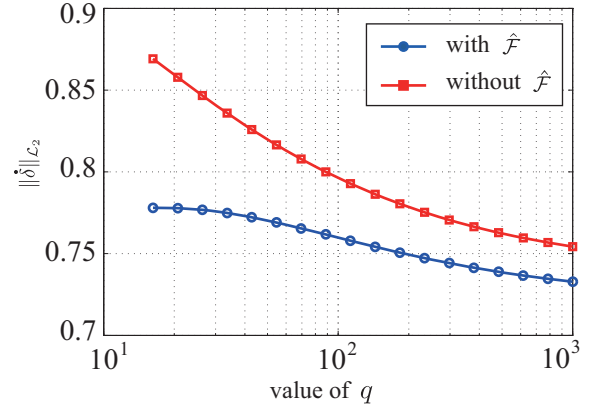


Fig. 6. Performance versus weight used for designing  $\hat{F}$  in the case with small  $x(0)$

sen from the ranges of  $[1, 100]$  and  $[1, 2]$ , respectively. Note that the dimension of the whole power network is 68.

Finally, we give the whole system dynamics in the form of (3). Let  $\zeta \in \mathbb{R}^{68}$  be the stacked version of  $\zeta_i$ , and

$$h(\zeta) = [h_1(\zeta), \dots, h_{23}(\zeta)]^\top, \quad h_i(\zeta) = \sum_{j \in \mathcal{N}_{i,j}} Y_{i,j} \sin(\delta_j - \delta_i)$$

and  $u := [u_1, \dots, u_5]$  be the inputs of all generators in  $\Sigma$ . Furthermore, let

$$\Xi = \text{diag}(A_i) + \text{diag}(r_i) \frac{\partial h}{\partial \zeta}(0),$$

$$g(\zeta) = \text{diag}(r_i) (h(\zeta) - \frac{\partial h}{\partial \zeta}(0) \zeta), \quad B = \text{diag}(b_i)_{i \in \{1, \dots, 5\}}.$$

Then, the whole system dynamics is described as

$$\dot{\zeta} = \Xi \zeta + g(\zeta) + [B^\top, 0]^\top u. \quad (29)$$

Note that  $\Xi$  represents the linearized dynamics of (29) around the origin because  $\frac{\partial g}{\partial \zeta}(0) = 0$ . In this setting, we take  $x \in \mathbb{R}^{30}$  and  $\xi \in \mathbb{R}^{38}$  as the part of  $\zeta$  associated with  $\Sigma$  and  $\mathcal{E}$  in Fig. 5, respectively. Then, we give  $A$  and  $f(\cdot)$  in (3) as the part of  $\Xi$  and  $g(\cdot)$  compatible with  $x$ . In addition, we give  $\mathcal{E}(\cdot)$  as the feedback from  $\xi$  to  $x$ . In what follows, we consider giving nonzero initial values for frequency differences of all generators and loads in  $\Sigma$  and for valve position differences of all generators in  $\Sigma$ .

### B. Demonstration of Retrofitting Control

We demonstrate the effectiveness of the proposed retrofitting control for the power network described in Section V-A. In Fig. 5, the red dotted lines show the trajectories of the frequency differences of all generators and loads for an initial state. To improve the convergence rate of the frequency differences, we construct a retrofit controller  $\mathcal{R}$  in (18), for which we take  $P = I_{30}$  and  $\hat{F}$  is designed by applying the LQR method for (5) minimizing the cost function

$$J = \int_0^\infty q \hat{x}^\top \hat{x} + u^\top u \, dt \quad (30)$$

with the scalar weight  $q = 10^3$ . Note that the dimension of  $\mathcal{R}$  coincides with that of  $x$ , i.e.,  $\hat{n} = 30$ . In Fig. 5, the

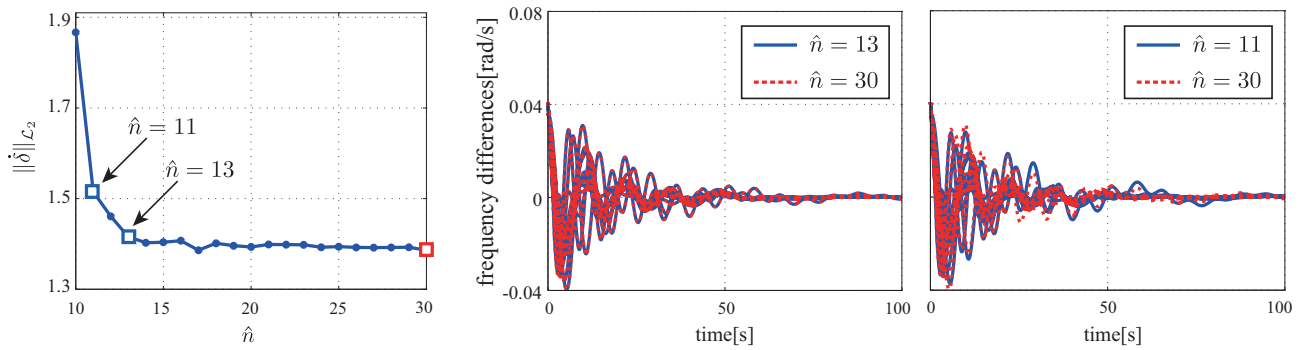


Fig. 8. Left: Performance versus dimension of retrofit controller. Middle and Right: Frequency differences of all generators and loads

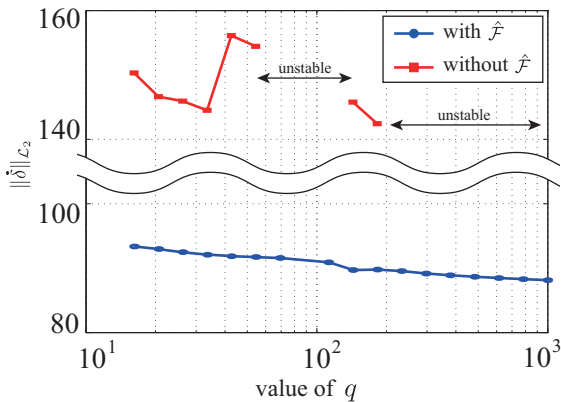


Fig. 7. Performance versus weight used for designing  $\hat{F}$  in the case with large  $x(0)$

blue solid lines show the case with  $\mathcal{R}$ . Furthermore, the  $\mathcal{L}_2$ -norm of the frequency differences of all nodes, denoted by  $\|\delta\|_{\mathcal{L}_2}$ , becomes 0.73, whereas  $\|\delta\|_{\mathcal{L}_2}$  without the controller is 5.24. These results imply that control performance of the whole network system is improved by implementing a retrofit controller.

Next, we investigate the relation between the value of  $\|\delta\|_{\mathcal{L}_2}$  and the feedback gain  $\hat{F}$  designed by the LQR method. To see this, we design several  $\hat{F}$  varying the weight  $q$  in (30), and calculate  $\|\delta\|_{\mathcal{L}_2}$  for each case. Let  $x(0)$  be small enough such that linear approximation of (3) is fine, i.e.,  $g(\cdot)$  in (29) is approximately negligible around the initial state. In this case,  $\|x(0)\| = 22.3$ . In Fig. 6, the blue line with circles corresponds to  $\|\delta\|_{\mathcal{L}_2}$  where the power network is controlled by the feedback control of (7). This result shows that the control performance, measured as  $\|\delta\|_{\mathcal{L}_2}$ , monotonically improves as the state weight  $q$  in (30) is made larger.

Furthermore, we investigate the effect stemming from the implementation of the compensator  $\hat{\mathcal{F}}(\cdot, \cdot)$  in (7). In Fig. 6, the red line with squares corresponds to  $\|\delta\|_{\mathcal{L}_2}$  in the case where  $\hat{\mathcal{F}}(\cdot, \cdot)$  in (7) is not used. We can see from this figure that the performance improves regardless of using the compensator  $\hat{\mathcal{F}}$  as the weight  $q$  is made larger. This is because the linear approximation of (3) is fine around

the initial state. Next, we take a larger initial state such that the linearized model does not well approximate (3). In this case,  $\|x(0)\| = 2236$ . In Fig. 7, we plot  $\|\delta\|_{\mathcal{L}_2}$  by the lines with circles and squares, where the same legends as those in Fig. 6 are used. Note that the cases without  $\hat{\mathcal{F}}$  in the ranges of  $q \in [10^{1.7}, 10^{2.1}]$  and  $q \in [10^{2.3}, 10^3]$  turn out to be unstable. From this figure, we see that the state feedback control without the compensator  $\hat{\mathcal{F}}(\cdot, \cdot)$  does not give any guarantee of control performance and destabilize the closed-loop system in some cases because the feedback controller designed based on the linearization does not work well around a region far from the origin. On the other hand, by using the compensator  $\hat{\mathcal{F}}(\cdot, \cdot)$ , we can theoretically guarantee the whole system stability in all cases and achieve the monotonicity of the performance improvement similar to Fig. 6. This is owing to the fact that the compensator  $\hat{\mathcal{F}}(\cdot, \cdot)$  prevents the stimulation to the nonlinear dynamics.

Moreover, we investigate the relation between the dimension of  $\mathcal{R}$  and the value of  $\|\delta\|_{\mathcal{L}_2}$ . To see this, we design several retrofit controllers varying their dimension  $\hat{n}$  along with the procedure shown in Section IV-B. In the leftmost in Fig. 8, we plot the resultant performance with respect to  $\hat{n}$ . We can see from this subfigure that the performance improves as  $\hat{n}$  increases. Furthermore, in the middle in Fig. 8, the blue solid lines show the transient responses of all generators and loads controlled by the retrofit controller with  $\hat{n} = 13$ . In addition, the red dotted lines correspond to the case of  $\hat{n} = 30$ . Note that no approximation is introduced if  $\hat{n} = 30$ . This subfigure implies that the control performance achieved by 13-dimensional retrofit controller is comparable with that by 30-dimensional controller. Finally, we plot the case with the 11-dimensional retrofit controller in the rightmost in Fig. 8, where we see a small degree of performance degradation.

## VI. CONCLUSION

In this paper, we have proposed a systematic method to design retrofit controllers for networked nonlinear systems. The retrofit controller aims at controlling a linear dynamics extracted from the subsystem of interest that can improve a degree of control performance for the subsystem, without spoiling the stability of the whole nonlinear networked

system. Towards systematic retrofit controller design, we have introduced a type of state-space expansion, called *hierarchical expansion*. The feature of the hierarchical expansion realization is its cascade structure, in which a linear dynamics extracted from the subsystem of interest is placed in the upstream part and the preexisting nonlinear dynamics is placed in the downstream part. By virtue of this cascade structure, the stabilization of each of upstream and downstream parts can be done independently. The effectiveness of the proposed method has been shown through a power network example.

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