Hierarchical Distributed Control for Networked Linear Systems

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***Abstract*— In this paper, we propose a design method of hierarchical distributed controllers for networked linear systems. The hierarchical distributed controller has an advantage that an $\mathcal{L}_2$-performance of the closed-loop system improves as improving a performance of local controllers that stabilize disjoint subsystems individually. Towards systematic design, we utilize state-space expansion that enables us to deal with the state variables associated with disjoint subsystems and those associated with interference among hierarchically clustered subsystems in a tractable manner. Moreover, by the integration of a hierarchical distributed observer having good compatibility with the structured controller, we build a framework to implement an observer-based hierarchical distributed control. The efficiency of the proposed control system is shown through an example of power networks.**

I. Introduction

As technology advances, the architecture of systems of interest to control community becomes more complex and larger in scale. For example, in smart grid, it is required to maintain supply-demand balance of power systems involving more than one million consumers with suitable control of a number of power plants [1], [2]. Typically, such large-scale complex systems are spatially distributed and networked. In view of this, it is crucial to build a framework for designing distributed control systems having good compatibility with the spatial distribution of networked systems [3], [4].

Even though many distributed controller synthesis methods have been developed in literature [5], these do not fully comply with practical application because controller design cannot be done in a distributed fashion. In view of this, a notion of *distributed design* is introduced in [6], where a performance limitation of controllers that are designed in a distributed manner is discussed by confining the class of systems to handle. In addition, a distributed design method in terms of the $\mathcal{L}_1$-induced norm has been developed for positive linear systems [7]. However, since this method fully utilizes a specific property of positive systems, generalization to a broader class of systems is not straightforward. One difficulty in such distributed design is that the improvement of local controller performance does not always imply that of closed-loop system performance due to negative interference among subsystems, and it may violate even the stability of closed-loop systems. Thus, it is important to understand how the variation of individual controllers in distributed control affects the performance as well as the stability of closed-loop systems.

Against this background, in this paper, we propose a design method of hierarchical distributed controllers for general linear systems. The hierarchical distributed controller has an advantage that an $\mathcal{L}_2$-performance of the closed-loop system is guaranteed for all sets of locally stabilizing controllers. Towards systematic distributed design, we first introduce state-space expansion, similar to one in [8], to independently deal with the state variables associated with disjoint subsystems and those associated with the interference among hierarchically clustered subsystems. This state-space expansion enables us to construct a hierarchically structured controller that attenuates the negative interference not only among hierarchically clustered subsystems but also among locally stabilizing controllers.

In [9], the authors have developed fundamental mathematical tools for designing hierarchical distributed controllers on the basis of state-space expansion. However, the authors have shown only the stability of closed-loop systems for all locally stabilizing controllers. In addition, we need to measure all outputs of subsystems.

On the other hand, in this paper, it is shown that an $\mathcal{L}_2$-performance of closed-loop systems improves as just improving an $\mathcal{L}_2$-performance of local controllers. This result clearly involves that in [9] as a special case. Moreover, by the integration of a hierarchical distributed observer proposed in [10], we build a framework to implement an observer-based hierarchical distributed control. The efficiency of the proposed control is shown through a numerical example for power networks.

The organization of this paper is as follows: In Section II, giving a mathematical formulation of hierarchical clustering of networked systems, we first formulate a design problem of hierarchical distributed controllers. In Section III-A, based on state-space expansion for dealing with networked systems in a hierarchical manner, we give a solution to the hierarchical distributed control design problem. Moreover, in Section III-B, we built a framework to implement an observer-based hierarchical distributed control by the integration of a hierarchical distributed observer. Then, Section IV devotes to show the efficiency of the proposed control structure through a numerical example of power networks. Finally, concluding remarks are provided in Section V.

Notation: We denote the set of real numbers by $\mathbb{R}$, the n-dimensional identity matrix by I_n, and the cardinality of a set $\mathcal{I}$ by $|\mathcal{I}|$. Furthermore, for $\mathcal{N} = \{1, \ldots, N\}$, we denote the block-diagonal matrix having matrices $M_1, \ldots, M_N$ on

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its diagonal blocks by $\mathrm{dg}(M_i)_{i\in\mathcal{N}}$. In particular, if not confusing, we omit the subscript of $i \in \mathcal{N}$. The $\mathcal{L}_2$-norm of a square integrable function $v(t) \in \mathbb{R}^n$ is defined by $\|v(t)\|_{\mathcal{L}_2} := \left(\int_0^\infty v^{\mathsf{T}}(t)v(t)dt\right)^{\frac{1}{2}}$. The $\mathcal{H}_\infty$-norm of a stable proper transfer matrix G is defined by $\|G(s)\|_{\mathcal{H}_\infty} := \sup_{\omega\in\mathbb{R}} \|G(j\omega)\|$ where $\|\cdot\|$ denotes the induced 2-norm.

II. Problem Formulation

A. Review of Decentralized Control

In this paper, we deal with networked linear systems composed of N subsystems. For each $i \in \mathcal{N} := \{1, \dots, N\}$, the dynamics of the ith subsystem is described by

$$\Sigma_i : \begin{cases} \dot{x}_i = A_i x_i + \sum_{j\neq i}^N A_{i,j}x_j + B_i u_i \\ y_i = C_i x_i \end{cases} \tag{1}$$

where $A_i \in \mathbb{R}^{n_i\times n_i}$, $A_{i,j} \in \mathbb{R}^{n_i\times n_j}$, $B_i \in \mathbb{R}^{n_i\times m_i}$, and $C_i \in \mathbb{R}^{p_i\times n_i}$. In this notation, we consider a set of local controllers that stabilizes each Σ_i by using the input signal u_i and the sensor signal y_i. The local controller associated with Σ_i is described by

$$\kappa_i : \begin{cases} \dot{\xi}_i = K_i\xi_i + L_i y_i \\ u_i = M_i\xi_i \end{cases} \tag{2}$$

where $K_i \in \mathbb{R}^{r_i\times r_i}$, $L_i \in \mathbb{R}^{r_i\times p_i}$, and $M_i \in \mathbb{R}^{m_i\times r_i}$.

Let us consider the disjoint subsystem with the local controller described by

$$\begin{bmatrix} \dot{x}_i \\ \dot{\xi}_i \end{bmatrix} = \begin{bmatrix} A_i & B_iM_i \\ L_iC_i & K_i \end{bmatrix} \begin{bmatrix} x_i \\ \xi_i \end{bmatrix}, \quad i\in\mathcal{N}. \tag{3}$$

For a given constant $\theta_i > 0$, we suppose that (3) satisfies

$$\|x_i(t)\|_{\mathcal{L}_2} \le \theta_i$$

for all $x_i(0) \in \mathbb{R}^{n_i}$ such that $\|x(0)\| = 1$ where $x := [x_1^{\mathsf{T}}, \dots, x_N^{\mathsf{T}}]^{\mathsf{T}}$. Obviously, if all subsystems are disjoint, then the closed-loop system $(\{\Sigma_i\}_{i\in\mathcal{N}}, \{\kappa_i\}_{i\in\mathcal{N}})$ has the $\mathcal{L}_2$-performance

$$\|x(t)\|_{\mathcal{L}_2} \le \|\theta\| \tag{4}$$

where $\theta := [\theta_1, \dots, \theta_N]^{\mathsf{T}}$. In what follows, we denote a set of local controllers achieving the $\mathcal{L}_2$-performance in (4) for disjoint subsystems, i.e., Σ_i in (1) with $A_{i,j} = 0$ for all $j \in \mathcal{N}\backslash\{i\}$, as $\{\kappa_i\}_{i\in\mathcal{N}} \in \mathcal{K}_\theta$.

However, if $A_{i,j} \neq 0$, i.e., if the subsystems are interconnected, the $\mathcal{L}_2$-performance of disjoint closed-loop systems does not provide any guarantee for the whole closed-loop system in general. In this paper, we consider designing a hierarchical distributed controller that attenuates negative interference among subsystems.

B. Hierarchical Distributed Control Problem

In what follows, we use the notation of

$$n := \sum_{i=1}^N n_i, \quad m := \sum_{i=1}^N m_i, \quad p := \sum_{i=1}^N p_i \quad r := \sum_{i=1}^N r_i$$

and

$$A := \begin{bmatrix} A_1 & \cdots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \cdots & A_N \end{bmatrix} \in \mathbb{R}^{n\times n}. \tag{5}$$

We consider introducing a hierarchical structure into networked systems. Let $\mathcal{L} := \{1, \dots, L\}$ with an integer L that represents the number of system layers. We define a family of index sets $\{\mathcal{N}^{(l)}\}_{l\in\mathcal{L}}$ such that

$$N \ge |\mathcal{N}^{(1)}| \ge \cdots \ge |\mathcal{N}^{(L)}| = 1, \quad \mathcal{N}^{(l)} = \{1, \dots, |\mathcal{N}^{(l)}|\}. \tag{6}$$

Furthermore, for each $l \in \{0, \dots, L-1\}$, we define a family of cluster sets $\{\mathcal{C}_i^{(l)}\}_{i\in\mathcal{N}^{(l+1)}}$ such that

$$\bigcup_{i\in\mathcal{N}^{(l+1)}} \mathcal{C}_i^{(l)} = \mathcal{N}^{(l)}, \quad \mathcal{C}_i^{(l)} \cap \mathcal{C}_j^{(l)} = \emptyset, \quad i \neq j, \tag{7}$$

where $\mathcal{N}^{(0)}$ is regarded as $\mathcal{N}$.

Let $A_i^{(l)} \in \mathbb{R}^{n_i^{(l)}\times n_i^{(l)}}$ denote the principal submatrix of A compatible with $\mathcal{C}_i^{(l-1)}$. By definition, it follows that

$$\sum_{i\in\mathcal{N}^{(l)}} n_i^{(l)} = n, \quad l \in \mathcal{L},$$

and $A^{(L)} = A$. In the rest of this paper, we regard $A_i^{(0)}$ as A_i for all $i \in \mathcal{N}$.

We give the dynamics of the whole networked system as

$$\Sigma : \begin{cases} \dot{x} = Ax + \mathrm{dg}(B_i)u + \sum_{l=1}^L \mathrm{dg}(B_i^{(l)})u^{(l)} \\ y = \mathrm{dg}(C_i)x \end{cases} \tag{8}$$

where the input signal $u := [u_1^{\mathsf{T}}, \dots, u_N^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^m$ and the output signal $y := [y_1^{\mathsf{T}}, \dots, y_N^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^p$ are used for the interconnection to local controllers, and the term of $u^{(l)}$ expresses an additional input signal from a hierarchical distributed controller to be explained below. In what follows, the pair $(A_i^{(l)}, B_i^{(l)})$, which is defined as being compatible with the hierarchical structure of networked systems, is assumed to be stabilizable for any $i \in \mathcal{N}^{(l)}$ and $l \in \mathcal{L}$. Similarly, for $\xi := [\xi_1^{\mathsf{T}}, \dots, \xi_N^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^r$, the dynamics of local controllers is expressed by

$$\{\kappa_i\}_{i\in\mathcal{N}} : \begin{cases} \dot{\xi} = \mathrm{dg}(K_i)\xi + \mathrm{dg}(L_i)(y+z) \\ u = \mathrm{dg}(M_i)\xi \end{cases} \tag{9}$$

where the term of z expresses an additional input signal as well.

To construct appropriate additional input signals $\{u^{(l)}\}_{l\in\mathcal{L}}$ and z, we consider designing a hierarchical distributed controller given by

$$\Phi^{(l)} : \begin{cases} \dot{\phi}^{(l)} = \mathrm{dg}(\mathbf{E}_i^{(l)})\phi^{(l)} + \mathbf{G}^{(l)}x + \sum_{k=l}^L \mathrm{dg}(\mathbf{B}_i^{(k)})u^{(k)} \\ u^{(l)} = \mathrm{dg}(\mathbf{F}_i^{(l)})(\phi^{(l)} - \phi^{(l+1)}) \end{cases} \tag{10}$$

where $\phi^{(L+1)}$ is regarded as zero, and

$$\mathbf{E}_i^{(l)} \in \mathbb{R}^{n_i^{(l)}\times n_i^{(l)}}, \quad \mathbf{F}_i^{(l)} \in \mathbb{R}^{m_i^{(l)}\times n_i^{(l)}}, \quad \mathbf{B}_i^{(l)} \in \mathbb{R}^{n_i^{(l)}\times m_i^{(l)}}$$
$$\mathbf{G}^{(l)} \in \mathbb{R}^{n\times n}$$

are design parameters. Moreover, the additional input to local controllers is given by

$$z = \mathrm{dg}(\mathbf{H}_i)\phi^{(1)}$$

where $\mathbf{H}_i \in \mathbb{R}^{p_i \times n_i}$ is another design parameter. In the rest of this paper, we suppose that $\xi(0) = 0$ and $\phi^{(l)}(0) = 0$ for all $l \in \mathcal{L}$. Furthermore, we denote the hierarchical distributed controller by $\{\Phi^{(l)}\}_{l\in\mathcal{L}}$.

In this setting, the following control problem for the closed-loop system $(\Sigma, \{\Phi^{(l)}\}_{l\in\mathcal{L}}, \{\kappa_i\}_{i\in\mathcal{N}})$ is addressed:

Problem 1: Given $\{\mathcal{N}^{(l)}\}_{l\in\mathcal{L}}$ and $\{\mathcal{C}_i^{(l)}\}_{i\in\mathcal{N}^{(l+1)}}$ such that (6) and (7), consider Σ in (8) with $\{\kappa_i\}_{i\in\mathcal{N}}$ in (9). Then, for a given constant $\epsilon > 0$, find $\{\Phi^{(l)}\}_{l\in\mathcal{L}}$ in (10) satisfying

$$\|x(t)\|_{\mathcal{L}_2} \leq \|\theta\| + \epsilon \tag{11}$$

for all $x(0) \in \mathbb{R}^n$ such that $\|x(0)\| = 1$ and $\{\kappa_i\}_{i\in\mathcal{N}} \in \mathcal{K}_\theta$.

In Problem 1, we formulate a problem to find a hierarchical distributed controller satisfying that an $\mathcal{L}_2$-performance of the closed-loop system, which necessarily implies the stability of the closed-loop system, is robustly guaranteed for all sets of locally stabilizing controllers.

III. Hierarchical Distributed Control Systems

A. Design of Hierarchical Distributed Controllers

To systematically design a hierarchical distributed controller, we consider transforming the realization of Σ into a tractable one based on the following state-space expansion:

Lemma 1: Given $\{\mathcal{N}^{(l)}\}_{l\in\mathcal{L}}$ and $\{\mathcal{C}_i^{(l)}\}_{i\in\mathcal{N}^{(l+1)}}$ such that (6) and (7), consider Σ in (8). For $l \in \mathcal{L}$, define

$$\begin{cases} \dot{\tilde{x}}^{(l)} = \mathrm{dg}(A_i^{(l)})\tilde{x}^{(l)} + \mathrm{dg}(B_i^{(l)})u^{(l)} + \Gamma^{(l)}\sum_{k=0}^{l-1}\tilde{x}^{(k)} \\ \dot{\tilde{x}}^{(0)} = \mathrm{dg}(A_i)\tilde{x}^{(0)} + \mathrm{dg}(B_i)u \end{cases} \tag{12}$$

where

$$\Gamma^{(l)} := \mathrm{dg}(A_i^{(l)})_{i\in\mathcal{N}^{(l)}} - \mathrm{dg}(A_i^{(l-1)})_{i\in\mathcal{N}^{(l-1)}}. \tag{13}$$

If $x(0) = \sum_{l=0}^{L} \tilde{x}_l(0)$, then

$$x(t) = \sum_{l=0}^{L} \tilde{x}_l(t), \quad t \geq 0 \tag{14}$$

for any u and $\{u^{(l)}\}_{l\in\mathcal{L}}$.

Proof: Let $\tilde{x} = [(\tilde{x}^{(L)})^{\mathsf{T}}, \ldots, (\tilde{x}^{(1)})^{\mathsf{T}}, (\tilde{x}^{(0)})^{\mathsf{T}}]^{\mathsf{T}}$. Noting that

$$T\tilde{A} = AT, \quad T\tilde{B} = \left[B^{(L)}, \ldots, \mathrm{dg}(B_i^{(1)}), \mathrm{dg}(B_i)\right]$$

for $T := [I_n, \ldots, I_n] \in \mathbb{R}^{n\times(L+1)n}$ where $\tilde{A}$ and $\tilde{B}$ are defined as in (15), we have $T\tilde{x}(t) = x(t)$. ■

Lemma 1 shows that the summation of all state variables of the expanded system in (12), which has a cascade structure shown in (15), coincides with the original state variable for any input signals. The cascade structure of (15) gives a clear insight into controlling the original system Σ in (8) by using input signals u and $\{u^{(l)}\}_{l\in\mathcal{L}}$. Based on this lemma, we have the following result:

Theorem 1: Given $\{\mathcal{N}^{(l)}\}_{l\in\mathcal{L}}$ and $\{\mathcal{C}_i^{(l)}\}_{i\in\mathcal{N}^{(l+1)}}$ such that (6) and (7), consider Σ in (8) with $\{\kappa_i\}_{i\in\mathcal{N}}$ in (9). Define $\{\Phi^{(l)}\}_{l\in\mathcal{L}}$ in (10) with

$$\begin{array}{lll} \mathbf{E}_i^{(l)} = \mathrm{dg}(A_j^{(l-1)})_{j\in\mathcal{C}_i^{(l-1)}}, & \mathbf{F}_i^{(l)} = F_i^{(l)}, & \mathbf{B}_i^{(l)} = B_i^{(l)} \\ \mathbf{G}^{(l)} = \sum_{k=l}^{L}\Gamma^{(k)}, & & \mathbf{H}_i = -C_i \end{array} \tag{16}$$

where $F_i^{(l)}$ satisfies that $A_i^{(l)} + B_i^{(l)}F_i^{(l)}$ is stable, and $\Gamma^{(l)}$ is defined as in (13). Furthermore, define

$$\gamma^{(l)} := \left\| \left(sI_n - \mathrm{dg}(A_i^{(l)} + B_i^{(l)}F_i^{(l)})\right)^{-1}\Gamma^{(l)} \right\|_{\mathcal{H}_\infty} \tag{17}$$

for each $l \in \mathcal{L}$. Then

$$\|x(t)\|_{\mathcal{L}_2} \leq \|\theta\| \prod_{l=1}^{L}\left(1 + \gamma^{(l)}\right) \tag{18}$$

for all $x(0) \in \mathbb{R}^n$ such that $\|x(0)\| = 1$ and $\{\kappa_i\}_{i\in\mathcal{N}} \in \mathcal{K}_\theta$.

Proof: Based on Lemma 1, we consider the state feedback of $u^{(l)} = \mathrm{dg}(F_i^{(l)})\tilde{x}^{(l)}, \quad l \in \mathcal{L}$, and the output feedback of

$$\begin{cases} \dot{\xi} = \mathrm{dg}(K_i)\xi + \mathrm{dg}(L_iC_i)\tilde{x}^{(0)} \\ u = \mathrm{dg}(M_i)\xi \end{cases}$$

for the expanded system in (12). By the coordinate transformation as

$$\phi^{(l)} = \sum_{k=l}^{L}\tilde{x}^{(k)}, \quad l \in \mathcal{L} \tag{19}$$

with the relation in (14), we have the autonomous system in (20) where

$$\begin{aligned} \Lambda^{(l)} &:= \mathrm{dg}\left(\mathrm{dg}(A_j^{(l-1)})_{j\in\mathcal{C}_i^{(l-1)}} + B_i^{(l)}F_i^{(l)}\right)_{i\in\mathcal{N}^{(l)}} \\ \Theta^{(l)} &:= \mathrm{dg}\left(B_i^{(l)}F_i^{(l)} - \mathrm{dg}(B_j^{(l-1)}F_j^{(l-1)})_{j\in\mathcal{C}_i^{(l-1)}}\right)_{i\in\mathcal{N}^{(l)}} \end{aligned}$$

with $B_i^{(0)}F_i^{(0)} = 0$. Giving $\tilde{x}^{(0)}(0) = x(0)$, we have $\|\tilde{x}^{(0)}(t)\|_{\mathcal{L}_2} \leq \|\theta\|$. Thus, (18) is proven by (14) in conjunction with the triangle inequality of the $\mathcal{L}_2$-norm. ■

Theorem 1 shows that the hierarchical distributed controller $\{\Phi^{(l)}\}_{l\in\mathcal{L}}$ given by (16), whose compositional units can be designed independently of designing local controllers, achieves the $\mathcal{L}_2$-performance as shown in (18). Thus, a solution to Problem 1 can be obtained by designing the feedback gains $F_i^{(l)}$ that make the values of $\gamma^{(l)}$ in (17) small enough. As shown in (18), the $\mathcal{L}_2$-performance of the whole closed-loop system improves as just improving the $\mathcal{L}_2$-performance of local controllers in (4).

From the structure of the transfer matrix in (17), we see that the function of the controller $\Phi^{(l)}$ is to attenuate negative interference among clustered subsystems, and the magnitude of interference attenuation is measured by $\gamma^{(l)}$. In addition, from $\mathbf{G}^{(l)}$ in (16), we notice that the lth layer controller $\Phi^{(l)}$ utilizes

$$w^{(l)} := \sum_{k=l}^{L}\Gamma^{(k)}x \tag{21}$$

as its input signal. Since $\Gamma^{(l)}$ in (13) represents the interconnection among clusters in the $(l-1)$th layer, the signal $w^{(l)}$

$$\tilde{A} := \begin{bmatrix} A^{(L)} & \Gamma^{(L)} & \cdots & \Gamma^{(L)} & \Gamma^{(L)} \\ & \mathrm{dg}(A_i^{(L-1)}) & \cdots & \Gamma^{(L-1)} & \Gamma^{(L-1)} \\ & & \ddots & \vdots & \vdots \\ & & & \mathrm{dg}(A_i^{(1)}) & \Gamma^{(1)} \\ & & & & \mathrm{dg}(A_i) \end{bmatrix}, \quad \tilde{B} := \begin{bmatrix} B^{(L)} & & & & \\ & \mathrm{dg}(B_i^{(L-1)}) & & & \\ & & \ddots & & \\ & & & \mathrm{dg}(B_i^{(1)}) & \\ & & & & \mathrm{dg}(B_i) \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} \dot{\phi}^{(L)} \\ \dot{\phi}^{(L-1)} \\ \vdots \\ \dot{\phi}^{(1)} \\ \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \Lambda^{(L)} & & & & \Gamma^{(L)} & \\ \Theta^{(L)} & \Lambda^{(L-1)} & & & \Gamma^{(L-1)} + \Gamma^{(L)} & \\ \vdots & \vdots & \ddots & & \vdots & \\ \Theta^{(L)} & \Theta^{(L-1)} & \cdots & \Lambda^{(1)} & \Gamma^{(1)} + \cdots + \Gamma^{(L)} & \\ \Theta^{(L)} & \Theta^{(L-1)} & \cdots & \Theta^{(1)} & A & \mathrm{dg}(B_i M_i) \\ & & & -\mathrm{dg}(L_i C_i) & \mathrm{dg}(L_i C_i) & \mathrm{dg}(K_i) \end{bmatrix} \begin{bmatrix} \phi^{(L)} \\ \phi^{(L-1)} \\ \vdots \\ \phi^{(1)} \\ x \\ \xi \end{bmatrix} \tag{20}$$

contains the information on the interaction among clustered subsystems. Consideration on the availability of $\{w^{(l)}\}_{l\in\mathcal{L}}$ will be given in Sections III-B below.

Remark 1: Note that $\Gamma^{(l)}$ becomes a lower-rank matrix if the interconnection among the corresponding clusters is sparser. Furthermore, the function of $\Phi^{(l)}$ is to attenuate negative interference among clustered subsystems, and the degree of interference attenuation is measured by $\gamma^{(l)}$ in (17). Thus, inspecting the structure of the transfer matrix in (17), we see that the magnitude of $\gamma^{(l)}$ can be efficiently reduced if

(i) the rank of $\Gamma^{(l)}$ is low enough, and
(ii) the input signal given by $B_i^{(l)}$ can effectively attenuate the interference signal injected through $\Gamma^{(l)}$.

Therefore, it is important to devise a method to appropriately find an actuator allocation as well as hierarchical clustering towards scalable implementation of hierarchical distributed control.

B. Integration with Hierarchical Distributed Observers

The hierarchical distributed controller $\{\Phi^{(l)}\}_{l\in\mathcal{L}}$ given by (16) requires the sensor signals $\{w^{(l)}\}_{l\in\mathcal{L}}$ in (21). In view of this, a number of sensors are possibly required to implement the control system. For example, to implement $\Phi^{(1)}$ in the first layer, we need to measure $w^{(1)}$ that contains the information on the interaction among *all* subsystems Σ_i.

To reduce the number of required sensors, on the basis of a hierarchical distributed observer [9] having good compatibility with the hierarchical structure of control systems, we consider estimating $w^{(l)}$ for lower layer controllers from other sensor signals. To this end, for $\hat{\mathcal{L}} := \{1,\ldots,\hat{L}\}$ with an integer $\hat{L} < L$, we suppose that

$$\begin{array}{ll} y^{(l)} := \mathrm{dg}(C_i^{(l)})x, & l \in \hat{\mathcal{L}} \\ v^{(l)} := \Gamma^{(l)}x, & l \in \mathcal{L}\backslash\hat{\mathcal{L}} \end{array} \tag{22}$$

are available as sensor signals from clustered subsystems. Furthermore, the pair $(A_i^{(l)}, C_i^{(l)})$ is supposed to be detectable for any $i \in \mathcal{N}^{(l)}$ and $l \in \hat{\mathcal{L}}$. Note that the availability of $\{v^{(l)}\}_{l\in\mathcal{L}\backslash\hat{\mathcal{L}}}$ is equal to that of $\{w^{(l)}\}_{l\in\mathcal{L}\backslash\hat{\mathcal{L}}}$. Under these supposition, we can achieve the following observer-based hierarchical distributed control:

Theorem 2: Given $\{\mathcal{N}^{(l)}\}_{l\in\mathcal{L}}$ and $\{\mathcal{C}_i^{(l)}\}_{i\in\mathcal{N}^{(l+1)}}$ such that (6) and (7), consider Σ in (8) with $\{\kappa_i\}_{i\in\mathcal{N}}$ in (9). For $y^{(l)}$ in (22) with $H_i^{(l)}$ such that $A_i^{(l)} - H_i^{(l)}C_i^{(l)}$ is stable, define $\{o^{(l)}\}_{l\in\hat{\mathcal{L}}}$ in (23) with

$$\hat{w}^{(l)} := \begin{cases} \sum_{k=l}^{\hat{L}} \hat{v}^{(k)} + w^{(\hat{L}+1)}, & l \in \hat{\mathcal{L}}, \\ w^{(l)}, & l \in \mathcal{L}\backslash\hat{\mathcal{L}}, \end{cases} \tag{24}$$

where $w^{(l)}$ is defined as in (21). Furthermore, by replacing $w^{(l)}$ with $\hat{w}^{(l)}$, define $\{\Phi^{(l)}\}_{l\in\mathcal{L}}$ in (10) with (16). Then $(\Sigma, \{\Phi^{(l)}\}_{l\in\mathcal{L}}, \{\kappa_i\}_{i\in\mathcal{N}})$ with $\{o^{(l)}\}_{l\in\hat{\mathcal{L}}}$ is stable for all $\{\kappa_i\}_{i\in\mathcal{N}} \in \mathcal{K}_\theta$.

Proof: Omit due to page limitation. ■

The hierarchical distributed observer $\{o^{(l)}\}_{l\in\hat{\mathcal{L}}}$ in (23) produces the estimate signals $\{\hat{w}^{(l)}\}_{l\in\hat{\mathcal{L}}}$, which represent the interaction among clustered subsystems, by using the available sensor signals in (22).

IV. Numerical Example

A. Power Network Model

In this section, we show the efficiency of the proposed hierarchical distributed control by an example of power networks. We deal with a power network model [1] composed of N subnetworks (subsystems), where the ith subsystem consists of n_i^{G} generators and n_i^{L} loads.

For $k \in \mathcal{N}_i^{\mathrm{G}} := \{1,\ldots,n_i^{\mathrm{G}}\}$, the dynamics of the kth generator is described by

$$\Sigma_{[i]k}^{\mathrm{G}} : \begin{cases} \dot{\zeta}_{[i]k} = A_{[i]k}^{\mathrm{G}}\zeta_{[i]k} + \frac{1}{M_{[i]k}^{\mathrm{G}}}b^{\mathrm{G}}\theta_{[i]k}^{\mathrm{G}} + \frac{1}{T_{[i]k}^{\mathrm{G}}}bu_{[i]k} \\ \delta_{[i]k}^{\mathrm{G}} = c^{\mathrm{G}}\zeta_{[i]k} \end{cases} \tag{26}$$

where each element of $\zeta_{[i]k} \in \mathbb{R}^3$ denotes a phase angle difference, an angular velocity difference and a mechanical input difference, and $\theta_{[i]k}^{\mathrm{G}} \in \mathbb{R}$, $\delta_{[i]k}^{\mathrm{G}} \in \mathbb{R}$, and $u_{[i]k} \in \mathbb{R}$ denote an electric output difference, a phase angle difference, and a valve position difference, respectively. Furthermore, the system matrices in (26) are given by

$$A_{[i]k}^{\mathrm{G}} := \begin{bmatrix} 0 & 1 & 0 \\ 0 & -D_{[i]k}^{\mathrm{G}}/M_{[i]k}^{\mathrm{G}} & -1/M_{[i]k}^{\mathrm{G}} \\ 0 & 0 & -1/T_{[i]k}^{\mathrm{G}} \end{bmatrix}$$

$$o^{(l)}:\begin{cases}\dot{\hat{x}}^{(l)}=\mathrm{dg}(A_i^{(l)}-H_i^{(l)}C_i^{(l)})\hat{x}^{(l)}+\mathrm{dg}(B_i)u+\sum_{k=1}^{L}\mathrm{dg}(B_i^{(k)})u^{(k)}+\mathrm{dg}(H_i^{(l)})y^{(l)}+\hat{w}^{(l+1)}\\ \hat{v}^{(l)}=\Gamma^{(l)}\hat{x}^{(l)}\end{cases} \tag{23}$$

$$\begin{aligned}A&=\mathrm{dg}\left(\mathrm{dg}(A^{\mathrm{G}}_{[i]k}),\mathrm{dg}(A^{\mathrm{L}}_{[i]k})\right)_{i\in\mathbb{N}}-\mathrm{dg}\left(\mathrm{dg}(\tfrac{1}{M^{\mathrm{G}}_{[i]k}}b^{\mathrm{G}}),\mathrm{dg}(\tfrac{1}{M^{\mathrm{L}}_{[i]k}}b^{\mathrm{L}})\right)_{i\in\mathbb{N}}Y\mathrm{dg}\left(\mathrm{dg}(c^{\mathrm{G}})_{k\in\mathbb{N}_i^{\mathrm{G}}},\mathrm{dg}(c^{\mathrm{L}})_{k\in\mathbb{N}_i^{\mathrm{L}}}\right)_{i\in\mathbb{N}}\\ B_i&=\begin{bmatrix}\mathrm{dg}(\frac{1}{T^{\mathrm{G}}_{[i]k}}b)\\ 0_{2n_i^{\mathrm{L}}\times n_i^{\mathrm{G}}}\end{bmatrix},\quad C_i=\begin{bmatrix}I_{n_i^{\mathrm{G}}}\otimes[I_2\ 0_{2\times1}] & 0_{2n_i^{\mathrm{G}}\times 2n_i^{\mathrm{L}}}\end{bmatrix}\end{aligned} \tag{25}$$

and $b^{\mathrm{G}}:=e_2^3,\quad c^{\mathrm{G}}:=(e_1^3)^{\mathsf{T}},\quad b:=e_3^3$ where $M^{\mathrm{G}}_{[i]k},\ D^{\mathrm{G}}_{[i]k}$ and $T^{\mathrm{G}}_{[i]k}$ denote a mechanical inertia, a damping coefficient and a turbine time constant, respectively, and $e_i^n\in\mathbb{R}^n$ denotes the ith column of I_n.

In a similar fashion, for $k\in\mathcal{N}_i^{\mathrm{L}}:=\{1,\ldots,n_i^{\mathrm{L}}\}$, the dynamics of the kth load is described by

$$\Sigma^{\mathrm{L}}_{[i]k}:\begin{cases}\dot{\psi}_{[i]k}&=A^{\mathrm{L}}_{[i]k}\psi_{[i]k}+\frac{1}{M^{\mathrm{L}}_{[i]k}}b^{\mathrm{L}}\theta^{\mathrm{L}}_{[i]k}\\ \delta^{\mathrm{L}}_{[i]k}&=c^{\mathrm{L}}\psi_{[i]k}\end{cases} \tag{27}$$

where each state of $\psi_{[i]k}\in\mathbb{R}^2$ denotes a phase angle difference and an angular velocity difference, and $\theta^{\mathrm{L}}_{[i]k}\in\mathbb{R}$ and $\delta^{\mathrm{L}}_{[i]k}\in\mathbb{R}$ denote an electric output difference and a phase angle difference, respectively. Furthermore, the system matrices in (27) are given by

$$A^{\mathrm{L}}_{[i]k}:=\begin{bmatrix}0&1\\0&-D^{\mathrm{L}}_{[i]k}/M^{\mathrm{L}}_{[i]k}\end{bmatrix},\quad b^{\mathrm{L}}:=e_2^2,\quad c^{\mathrm{L}}:=(e_1^2)^{\mathsf{T}}$$

where $M^{\mathrm{L}}_{[i]k}$ and $D^{\mathrm{L}}_{[i]k}$ denote positive constants that represent an inertia constant and a damping coefficient, respectively. The interconnection structure among generators and loads are given by

$$\theta=-Yy,\quad\begin{cases}\theta:=[\,(\theta_1^{\mathrm{G}})^{\mathsf{T}},(\theta_1^{\mathrm{L}})^{\mathsf{T}},\cdots,(\theta_N^{\mathrm{G}})^{\mathsf{T}}(\theta_N^{\mathrm{L}})^{\mathsf{T}}\,]^{\mathsf{T}}\\ \delta:=[\,(\delta_1^{\mathrm{G}})^{\mathsf{T}},(\delta_1^{\mathrm{L}})^{\mathsf{T}},\cdots,(\delta_N^{\mathrm{G}})^{\mathsf{T}}(\delta_N^{\mathrm{L}})^{\mathsf{T}}\,]^{\mathsf{T}}\end{cases} \tag{28}$$

where $Y\in\mathbb{R}^{N_Y\times N_Y}$ represents an admittance matrix satisfying

$$Y=Y^{\mathsf{T}},\quad Y\mathbf{1}_{N_Y}=0,\quad\begin{cases}\mathbf{1}_n&:=[1,\ldots,1]^{\mathsf{T}}\in\mathbb{R}^n\\ N_Y&:=\sum_{i=1}^{N}n_i^{\mathrm{G}}+n_i^{\mathrm{L}},\end{cases}$$

and

$$\begin{matrix}\theta_i^{\star}:=[\theta^{\star}_{[i]1},\cdots,\theta^{\star}_{[i]n_i^{\star}}]\\ \delta_i^{\star}:=[\delta^{\star}_{[i]1},\cdots,\delta^{\star}_{[i]n_i^{\star}}]\end{matrix},\quad\star\in\{\mathrm{G},\mathrm{L}\}.$$

We define a state variable as $x:=[\zeta_1^{\mathsf{T}},\psi_1^{\mathsf{T}},\ldots,\zeta_N^{\mathsf{T}},\psi_N^{\mathsf{T}}]^{\mathsf{T}}$ where $\zeta_i:=[\zeta^{\mathsf{T}}_{[i]1},\cdots,\zeta^{\mathsf{T}}_{[i]n_i^{\mathrm{G}}}]^{\mathsf{T}}$ and $\psi_i:=[\psi^{\mathsf{T}}_{[i]1},\cdots,\psi^{\mathsf{T}}_{[i]n_i^{\mathrm{L}}}]^{\mathsf{T}}$. Furthermore, we define the input u in (8) by $u:=[u^{\mathsf{T}}_{[1]},\cdots,u^{\mathsf{T}}_{[N]}]^{\mathsf{T}}$, $u_{[i]}:=[u_{[i]1},\cdots,u_{[i]n_i^{\mathrm{G}}}]$ and the output y by

$$\begin{aligned}y&:=[(\zeta^{1:2}_{[1]})^{\mathsf{T}},\cdots,(\zeta^{1:2}_{[N]})^{\mathsf{T}}]^{\mathsf{T}}\\ \zeta^{1:2}_{[i]}&:=[(\zeta^{1:2}_{[i]1})^{\mathsf{T}},\cdots,(\zeta^{1:2}_{[i]n_i^{\mathrm{G}}})^{\mathsf{T}}]^{\mathsf{T}}\end{aligned} \tag{29}$$

where $\zeta^{1:2}_{[i]k}\in\mathbb{R}^2$ denotes the first and second elements of $\zeta_{[i]k}\in\mathbb{R}^3$. In this notation, for Σ in (8), the system matrices of the whole power network is given by (25), where $\otimes$ and $0_{n\times m}\in\mathbb{R}^{n\times m}$ denote the Kronecker product and the zero matrix, respectively.

Finally, we consider giving additional input ports used by a hierarchical distributed controller. Supposing that several generators have the ports for controllers in the lth layer, we give $B^{(l)}$ in (8) as a matrix composed of a part of columns of $\mathrm{dg}(B_i)_{i\in\mathcal{N}}$. Similarly to this, we give the additional output $y^{(l)}$ in (22) for observers in the lth layer as a part of y in (29).

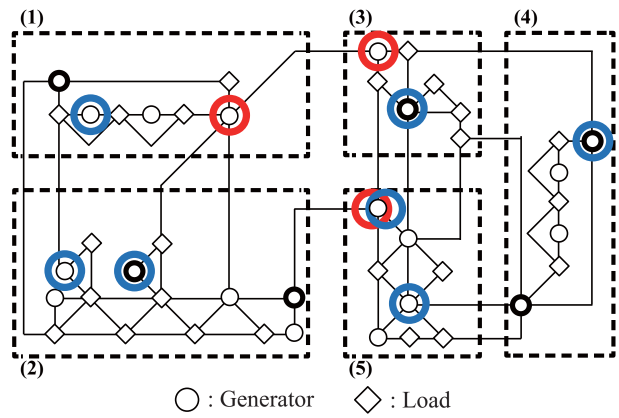

Fig. 1. Interconnection structure among generators and loads.

B. Hierarchical Distributed Control of Power Networks

In what follows, we implement the hierarchical distributed control to a power network composed of five subsystems. The interconnection structure among generators and loads is shown in Fig. 1, where generators and loads are denoted by circles and diamonds, respectively. The blue and red circles represent the generators having additional input ports and sensors to be used for a hierarchical distributed controller and observer.

In this power network model, 20 generators and 24 loads are interconnected. Thus, the interconnected system is 108-dimensional, i.e., $n=108$. For generators and loads, the parameters $M^{\mathrm{G}}_{[i]k}$, $D^{\mathrm{G}}_{[i]k}$, $T^{\mathrm{G}}_{[i]k}$, $M^{\mathrm{L}}_{[i]k}$ and $D^{\mathrm{L}}_{[i]k}$ are randomly chosen from $\{10,90\}$, $\{0.1,0.4\}$, $\{3.0,10\}$, $\{5,10,30\}$ and $\{0.1,0.3,0.5\}$, respectively. Furthermore, the elements of Y in (28) compatible with subsystem interconnection are given as 1, and those compatible with interconnection inside the subsystems are randomly chosen from $[0.1,1.0]$. In what follows, we consider a situation where the frequency of the power system suddenly varies. To simulate this, we give nonzero initial values for the angular velocity of generators, i.e., $\dot{\delta}^{\mathrm{G}}_{[i]k}(0)\neq0$.

First, we design a set of locally stabilizing controllers $\{\kappa_i\}_{i\in\mathcal{N}}$ in (9) by the LQR design techniques. By changing the weighting parameters for the LQR design, we obtain three sets of locally stabilizing controllers. The resultant values of $\|\theta\|$ in (18) are 1388, 423 and 126, respectively. To see the behavior of the closed-loop system, we show its initial

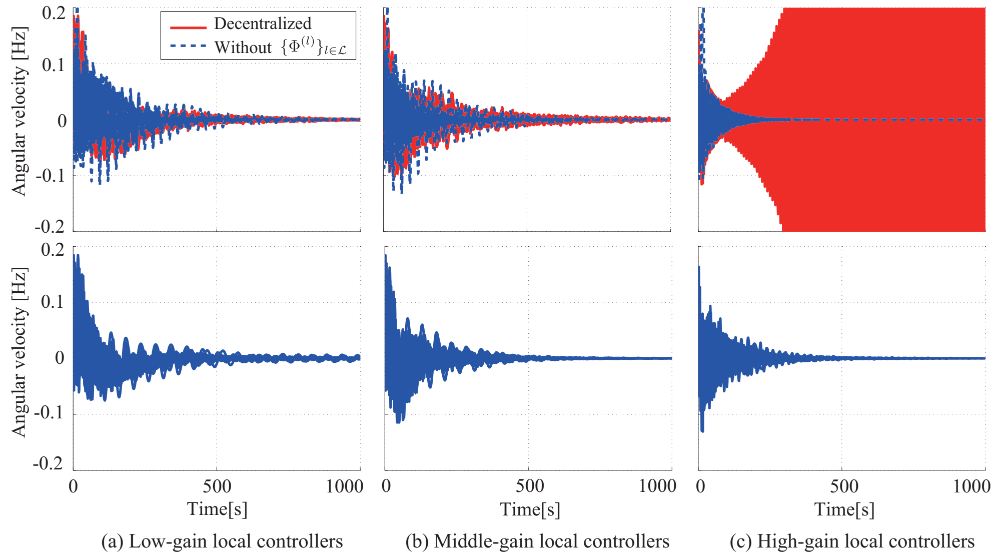

Fig. 2. Initial value responses of power network model.

value responses in the upper half of Fig. 2 (a)-(c), where the angular velocities of all generators and loads are shown. In this figure, the trajectories of disjoint subsystems, i.e., the system without subsystem interconnection, are shown by the dashed lines, and those of subsystems with interconnection are shown by the solid lines. From this result, we see that, even though the convergence rate of the system without subsystem interconnection becomes higher as improving the $\mathcal{L}_2$-performance of local controllers, the instability of the closed-loop system is induced by negative interference among subsystems.

Next, using a hierarchical distributed controller and observer, we aim at improving the $\mathcal{L}_2$-performance of the whole closed-loop system. Let $L = 2$, and we give a family of cluster sets as $\mathcal{C}_1^{(0)} = \{1, 2\}$, $\mathcal{C}_2^{(0)} = \{3, 4, 5\}$, $\mathcal{C}_1^{(1)} = \{1, 2\}$ where each of $\mathcal{C}_1^{(0)}$ and $\mathcal{C}_2^{(0)}$ includes 10 generators and 12 loads. Furthermore, in Fig. 1, we represent generators having additional input ports used for controllers in the first and second layers by the blue and red circles, respectively. In addition, by letting $\hat{L} = 1$ in (22), which means that the interaction among the five subsystems is estimated by an observer, we attach additional sensors for a hierarchical distributed observer on the generators represented by thick black circles in Fig. 1.

We design each controller $\Phi^{(l)}$ by minimizing $\gamma^{(l)}$ in (17). In addition, inspecting the structure of hierarchical distributed observers in (23), we design the observer gain $H_i^{(l)}$ by minimizing the $\mathcal{H}_\infty$-norm of the transfer matrix compatible with the pair $(\mathrm{dg}(A_i^{(l)} - H_i^{(l)} C_i^{(l)}), \Gamma^{(l)})$. Then, we implement the hierarchical distributed controller and observer. In the lower half of Fig. 2 (a)-(c), we show the initial value responses of the hierarchical distributed control system. For each case of (a)-(c), the resultant value of $\|x\|_{\mathcal{L}_2}$ is 1189, 491 and 234, respectively. From this result, we see that the $\mathcal{L}_2$-performance of the closed-loop system improves as improving the performance of local controllers, owing to the attenuation of interference among subsystems.

V. Conclusion

In this paper, we have proposed a design method of hierarchical distributed controllers for networked linear systems. For systematic design, we use state-space expansion that enables us to construct a hierarchically structured controller that attenuates negative interference among hierarchically clustered subsystems as well as among locally stabilizing controllers. Furthermore, by the integration of a hierarchical distributed observer, we have built a framework to implement an observer-based hierarchical distributed control. Finally, the efficiency of the proposed method has been shown through an illustrative example of power networks.

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