Low-Dimensional Functional Observer Design for Linear Systems via Observer Reduction Approach

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Abstract— **This paper proposes a design method for low-dimensional linear functional observers based on a model reduction approach. In contrast to existing methods for designing approximate observers, this method can guarantee not only stability preservation but also an a priori $\mathcal{L}_2$-error bound for the observer approximation. Moreover, owing to the fact that this method can be applied to any Luenberger-type functional observers, the method is compatible with the standard feedback gain determination methods, such as pole placement techniques. The efficiency of the proposed method is shown through a numerical example for electric power network systems.**

I. Introduction

Along with the recent technical development in engineering, the architecture of systems covered by the control community has tended to become more complex and larger in scale. For example, in smart grid, it is required to control an electric power system which involves hundreds of thousands consumers. In order to control such large-scale systems, observers are needed to estimate the states of systems. This is simply because it is difficult to measure all states. However, since existing observer is comparably large with a system to be observed, a high-dimensional observer is required for large-scale systems. In view of this, a design method for low-dimensional observers is crucial to deal with large-scale systems.

For linear systems, a number of observer design methods have been developed [1], [2], [3], [4], [5]. In this line of work, full or partial state observers can be designed from the view point of *exactly* canceling the effect of external input signals with respect to state estimation error. In other words, the notion of *approximation* is not introduced in these design methods. This implies that the state-space of observers must include states having even little influence on state estimation. Therefore, it is difficult to design low-dimensional observers based on the above methods.

Furthermore, low-dimensional observer design methods have been addressed in [6], [7], [8]. For example, [8] proposes a design procedure of reduced order observers for approximated models obtained by the balanced truncation. However, in such observer design, it is not necessarily easy to obtain a reduced order observer with a specified error precision in a direct way. This is because the performance of the observation is indirectly determined throughout the approximation error by model reduction.

In this paper, we propose a novel method of low-dimensional observer design that satisfies a specified estimation error precision. The proposed approach and the approaches mentioned above are schematically depicted in Fig. 1. In contrast to the existing approaches, our approach uses observer reduction where a reduced order observer approximates a full-state observer for the original system. A major advantage of this approach is that model reduction techniques are available. In particular, we utilize structured model reduction proposed in [9], [10], where an a priori approximation error bound is obtained. As a result, we derive an a priori $\mathcal{L}_2$-error bound on the performance degradation for the observer reduction. It should be remarked that, in the proposed design method, a feedback gain of the full state observer can be designed independently of reduction. Thus, the proposed method can use existing design methods to determine the gain. Finally, the efficiency of the proposed method is shown through an example of electric power network systems.

This paper is organized as follows: In section II, we formulate a design problem of low-dimensional functional observers. In section III, we devise a methodology to design the observer by biorthogonal projection. Furthermore, an a priori state estimation error bound is also derived. Section IV shows the efficiency of the proposed method by applying it to an electric power network. Finally, section V concludes this paper.

Notation The following notation is to be used:

$\mathbb{R}$	set of real numbers
I_n	unit matrix of size $n \times n$
$M \prec O_n$ ($M \preceq O_n$)	negative (semi)definiteness of a symmetric matrix $M \in \mathbb{R}^{n\times n}$
$M \succ O_n$ ($M \succeq O_n$)	positive (semi)definiteness of a symmetric matrix $M \in \mathbb{R}^{n\times n}$
$\mathrm{im}(M)$	range space spanned by the column vectors of a matrix M
$\mathrm{tr}(M)$	trace of a matrix M
$\mathrm{diag}(M_1, \ldots, M_n)$	block diagonal matrix having matrices $M_1, \ldots, M_n$ on its block diagonal

The $\mathcal{L}_2$-norm of a square integrable function $v(t) \in \mathbb{R}^n$ is defined by

$$\|v(t)\|_{\mathcal{L}_2} := \left(\int_0^\infty v(t)^{\mathsf{T}} v(t) dt \right)^{\frac{1}{2}}.$$

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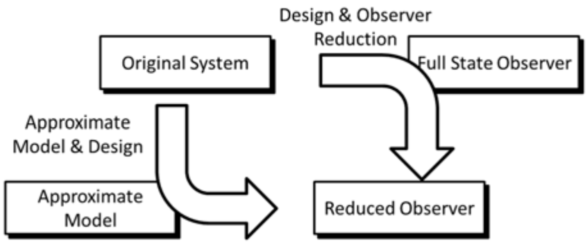

Fig. 1. Schematic depiction toward low-dimensional observers.

The $\mathcal{H}_\infty$-norm of a stable proper transfer matrix G and the $\mathcal{H}_2$-norm of a stable strictly proper transfer matrix G are respectively defined by

$$\|G(s)\|_{\mathcal{H}_\infty} := \sup_{\omega\in\mathbb{R}} \|G(j\omega)\|,$$
$$\|G(s)\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}(G(j\omega)G^{\mathsf{T}}(-j\omega))d\omega\right)^{\frac{1}{2}}$$

where $\|\cdot\|$ denotes the induced 2-norm.

II. Problem Formulation

A. Design Problem of Low-dimensional Functional Observer

In this section, we formulate a design problem for low-dimensional observers. Let us consider an n-dimensional linear system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \\ z = Sx \end{cases} \tag{1}$$

with $x(0) = x_0$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m_u}$, $C \in \mathbb{R}^{m_y\times n}$, $D \in \mathbb{R}^{m_y\times m_u}$ and $S \in \mathbb{R}^{m_z\times n}$. In this system, $y \in \mathbb{R}^{m_y}$ denotes a measurement output signal and $z \in \mathbb{R}^{m_z}$ denotes a signal to be estimated, which can be regarded as a specific sets of states of interest. In this paper, to simplify the arguments, we deal with only stable systems, with similar results available also for unstable systems, and assume that the system is observable, i.e., the observability matrix $[C^{\mathsf{T}}, (CA)^{\mathsf{T}}, \ldots, (CA^{n-1})^{\mathsf{T}}]^{\mathsf{T}}$ has full rank.

For Σ in (1), we define an $\hat{n}$-dimensional functional observer by

$$\hat{\Sigma}_{\mathrm{obs}} : \begin{cases} \dot{\hat{\xi}} \ = \hat{A}\hat{\xi} + \hat{B}u + \hat{H}(y - \hat{y}_{\mathrm{obs}}) \\ \hat{y}_{\mathrm{obs}} = \hat{C}\hat{\xi} + \hat{D}u \\ \hat{z}_{\mathrm{obs}} = \hat{S}\hat{\xi} \end{cases} \tag{2}$$

with $\hat{\xi}(0) = \hat{\xi}_0$, $\hat{A} \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $\hat{B} \in \mathbb{R}^{\hat{n}\times m_u}$, $\hat{C} \in \mathbb{R}^{m_y\times\hat{n}}$, $\hat{D} \in \mathbb{R}^{m_y\times m_u}$, $\hat{S} \in \mathbb{R}^{m_z\times\hat{n}}$ and $\hat{H} \in \mathbb{R}^{\hat{n}\times m_y}$. In what follows, $\hat{\Sigma}_{\mathrm{obs}}$ is called a functional observer since the estimated signal $\hat{z}_{\mathrm{obs}}$ is a function of the state $\hat{\xi}$. Without loss of generality, we focus on the case of $\hat{n} \le n$. In this notation, a design problem of low-dimensional functional observers is defined as follows: Given an n-dimensional linear system Σ in (1), find an $\hat{n}$-dimensional functional observer $\hat{\Sigma}_{\mathrm{obs}}$ in (2) such that $\hat{z}_{\mathrm{obs}}$ estimates z in a suitable sense.

One approach to this problem is to design an observer for approximate models (low-dimensional models) of Σ in (1), which can be obtained by applying existing model reduction methods; see, e.g., [8], [11]. However, this approach possibly leads to an undesirable result, since the performance of the observation may be degraded by even a small error in the approximation of Σ. In this sense, for designing a desirable low-dimensional observer, we should deal with an approximation error by explicitly taking into account the dynamics of both Σ and $\hat{\Sigma}_{\mathrm{obs}}$.

B. Problem Reformulation via Observer Reduction

In this subsection, we reformulate the above design problem as an observer reduction problem. For Σ in (1), let us consider the following Luenberger-type functional observer

$$\Sigma_{\mathrm{obs}} : \begin{cases} \dot{\xi} \ = A\xi + Bu + H(y - y_{\mathrm{obs}}) \\ y_{\mathrm{obs}} = C\xi + Du \\ z_{\mathrm{obs}} = S\xi \end{cases} \tag{3}$$

with $\xi(0) = \xi_0$. In the rest of this paper, we assume that the feedback gain $H \in \mathbb{R}^{n\times m_y}$ is designed such that the trajectory of the observation error

$$e_z := z - z_{\mathrm{obs}} \tag{4}$$

is desirable.

Let us construct a reduced order functional observer by approximating Σ_{obs} in (3). More specifically, using biorthogonal projection [12], we construct $\hat{\Sigma}_{\mathrm{obs}}$ in (2) by

$$\begin{aligned} &\hat{A} = PAP^\dagger, \quad \hat{B} = PB, \quad \hat{C} = CP^\dagger \\ &\hat{D} = D, \quad \hat{H} = PH, \quad \hat{S} = SP^\dagger \end{aligned} \tag{5}$$

where $P \in \mathbb{R}^{\hat{n}\times n}$ and $P^\dagger \in \mathbb{R}^{n\times\hat{n}}$ satisfy $PP^\dagger = I_{\hat{n}}$ and $\hat{\xi}_0 = P\xi_0$. We define the observation error for this reduced observer by

$$\hat{e}_z := z - \hat{z}_{\mathrm{obs}}. \tag{6}$$

In this notation, we address the following observer reduction problem.

Problem 1: Given a linear system Σ in (1), find an $\hat{n}$-dimensional functional observer $\hat{\Sigma}_{\mathrm{obs}}$ in (2) described by the system matrices (5) such that $\hat{n} \le n$ holds and the performance degradation

$$\|e_z - \hat{e}_z\|_{\mathcal{L}_2} \tag{7}$$

is small enough for the impulse input $u \in \mathbb{R}^{m_u}$ and any bounded initial condition $x_0 \in \mathbb{R}^n$, where $e_z \in \mathbb{R}^{m_z}$ and $\hat{e}_z \in \mathbb{R}^{m_z}$ are defined as in (4) and (6).

Note that, for a given system Σ with a feedback gain H, finding $\hat{\Sigma}_{\mathrm{obs}}$ just coincides with finding the biorthogonal projection described by P and $P^\dagger$ in (5). In the following section, we analyze how the choice of P and $P^\dagger$ affects the property of the performance degradation $\|e_z - \hat{e}_z\|_{\mathcal{L}_2}$ in (7).

III. Main Results

A. Analysis of Functional Observer Reduction

In this subsection, we investigate the relation between the performance degradation $\|e_z - \hat{e}_z\|_{\mathcal{L}_2}$ and the choice of the biorthogonal projection defined by P and $P^\dagger$. First, we show the following lemma that will be needed for an error analysis below.

Lemma 1: Let a stable linear system Σ in (1) be given, and suppose that there exists $V = V^\mathsf{T} \succ \mathcal{O}_n$ such that

$$\mathcal{S}_\gamma(A, S; V) \prec \mathcal{O}_{2n} \tag{8}$$

holds for

$$\mathcal{S}_\gamma(A, S; V) := \begin{bmatrix} A^\mathsf{T} V + VA + S^\mathsf{T} S & VA \\ A^\mathsf{T} V & -\gamma^2 V \end{bmatrix}. \tag{9}$$

Let a Cholesky factor $V_{\frac{1}{2}}$ of V such that $V = V_{\frac{1}{2}}^\mathsf{T} V_{\frac{1}{2}}$, and define $P = W V_{\frac{1}{2}}$ and $P^\dagger = V_{\frac{1}{2}}^{-1} W^\mathsf{T}$ for $W \in \mathbb{R}^{\hat{n}\times n}$ such that $WW^\mathsf{T} = I_{\hat{n}}$ holds. Then

$$\left\| SP^\dagger (sI_{\hat{n}} - PAP^\dagger)^{-1} PAV_{\frac{1}{2}}^{-1} \right\|_{\mathcal{H}_\infty} < \gamma \tag{10}$$

holds for any W.

Proof: We use the bounded real lemma [12] to prove (10). To this end, it suffices to show that

$$\mathcal{F}_\gamma(PAP^\dagger, PAV_{\frac{1}{2}}^{-1}, SP^\dagger; I_{\hat{n}}) \prec \mathcal{O}_{n+\hat{n}} \tag{11}$$

holds, where

$$\mathcal{F}_\gamma(A, B, C; V) := \begin{bmatrix} A^\mathsf{T} V + VA + C^\mathsf{T} C & VB \\ B^\mathsf{T} V & -\gamma^2 I_{m_u} \end{bmatrix}.$$

The left hand side of (11)

$$\mathcal{F}_\gamma(PAP^\dagger, PAV_{\frac{1}{2}}^{-1}, SP^\dagger; I_{\hat{n}})$$

can be rewritten as

$$\tilde{W} \mathcal{S}_\gamma(V_{\frac{1}{2}} A V_{\frac{1}{2}}^{-1}, SV_{\frac{1}{2}}^{-1}; I_n) \tilde{W}^\mathsf{T}$$

where $\tilde{W} := \mathrm{diag}(W, W)$. Note that W has full row rank. On the other hand, we notice that $\mathcal{S}_\gamma(A, S; V)$ can be rewritten as

$$\tilde{V}^\mathsf{T} \mathcal{S}_\gamma(V_{\frac{1}{2}} A V_{\frac{1}{2}}^{-1}, SV_{\frac{1}{2}}^{-1}; I_n) \tilde{V}$$

where $\tilde{V} := \mathrm{diag}(V_{\frac{1}{2}}, V_{\frac{1}{2}})$. Thus, (8) is equivalent to

$$\mathcal{S}_\gamma(V_{\frac{1}{2}} A V_{\frac{1}{2}}^{-1}, SV_{\frac{1}{2}}^{-1}; I_n) \prec \mathcal{O}_{2n}. \tag{12}$$

Pre- and post-multiplication of (12) by $\tilde{W}$ yields (11). Thus, (10) follows. ∎

Lemma 1 shows that, if there exists a positive definite matrix V such that (8) holds, then $PAP^\dagger$ is stable and a projection-based reduced model admits the $\mathcal{H}_\infty$-bound shown in (10) for any W. In fact, this $\mathcal{H}_\infty$-bound plays an important role for the error analysis in the following theorem, which gives a solution to the observer reduction problem defined in Section II-B. In what follows, we assume that $\xi_0 = 0$ for Σ_{obs} in (3) for simplicity.

Theorem 1: Given Σ in (1) and Σ_{obs} in (3), define

$$\mathcal{A} = \begin{bmatrix} A - HC & HC \\ 0 & A \end{bmatrix}, \ \mathcal{B} := \begin{bmatrix} B \\ B \end{bmatrix}. \tag{13}$$

Let $\mathcal{K} = \mathcal{K}^\mathsf{T} \succeq \mathcal{O}_n$ such that

$$\mathcal{A}\mathcal{K} + \mathcal{K}\mathcal{A}^\mathsf{T} + \mathcal{B}\mathcal{B}^\mathsf{T} + \alpha^2 \mathrm{diag}(0, I_n) = 0 \tag{14}$$

holds for $\alpha \geq \|x_0\|$. Furthermore, let $\gamma > 0$ such that

$$\mathcal{S}_\gamma(A - HC, S; V) \prec \mathcal{O}_{2n} \tag{15}$$

holds for $\mathcal{S}_\gamma$ defined as in (9), and define a Cholesky factor $V_{\frac{1}{2}}$ of V such that $V = V_{\frac{1}{2}}^\mathsf{T} V_{\frac{1}{2}}$. If $W \in \mathbb{R}^{\hat{n}\times n}$ satisfies $WW^\mathsf{T} = I_{\hat{n}}$ and

$$\mathrm{im}\left((SV_{\frac{1}{2}}^{-1})^\mathsf{T}\right) \subseteq \mathrm{im}(W^\mathsf{T}), \ \sqrt{\mathrm{tr}(\Phi) - \mathrm{tr}(W\Phi W^\mathsf{T})} \leq \epsilon \tag{16}$$

where

$$\Phi := V_{\frac{1}{2}} \mathcal{K}_{1:n} V_{\frac{1}{2}}^\mathsf{T} \in \mathbb{R}^{n\times n} \tag{17}$$

and $\mathcal{K}_{1:n} \in \mathbb{R}^{n\times n}$ denotes the principal submatrix of $\mathcal{K}$ corresponding to the first n rows and columns, then $\hat{\Sigma}_{\mathrm{obs}}$ in (2) described by (5) with $P = WV_{\frac{1}{2}}$ and $P^\dagger = V_{\frac{1}{2}}^{-1} W^\mathsf{T}$ satisfies

$$\|e_z - \hat{e}_z\|_{\mathcal{L}_2} \leq \gamma\epsilon \tag{18}$$

for the impulse input u, where e_z and $\hat{e}_z$ are defined as in (4) and (6), respectively.

Proof: Define $\mathcal{S} := [-S, S]$, $\mathcal{X}_0 := [0, x_0^\mathsf{T}]^\mathsf{T}$ and

$$\mathcal{P} := \mathrm{diag}(P, I_n), \quad \mathcal{P}^\dagger := \mathrm{diag}(P^\dagger, I_n).$$

By letting $\mathcal{X} := [\xi^\mathsf{T}, x^\mathsf{T}]^\mathsf{T}$, we have

$$\Sigma_{e_z} : \begin{cases} \dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + \mathcal{B}u \\ e_z = \mathcal{S}\mathcal{X} \end{cases}$$

with $\mathcal{X}(0) = \mathcal{X}_0$. Similarly, by letting $\hat{\mathcal{X}} := [\hat{\xi}^\mathsf{T}, x^\mathsf{T}]^\mathsf{T}$, we have

$$\hat{\Sigma}_{\hat{e}_z} : \begin{cases} \dot{\hat{\mathcal{X}}} = \mathcal{P}\mathcal{A}\mathcal{P}^\dagger \hat{\mathcal{X}} + \mathcal{P}\mathcal{B}u \\ \hat{e}_z = \mathcal{S}\mathcal{P}^\dagger \hat{\mathcal{X}} \end{cases}$$

with $\hat{\mathcal{X}}(0) = \mathcal{P}\mathcal{X}_0$. Consider the similarity transformation of the error system defined by Σ_{e_z} and $\hat{\Sigma}_{\hat{e}_z}$ with

$$T = \begin{bmatrix} -\mathcal{P} & I_{n+\hat{n}} \\ I_{2n} & 0 \end{bmatrix}, \ T^{-1} = \begin{bmatrix} 0 & I_{2n} \\ I_{n+\hat{n}} & \mathcal{P} \end{bmatrix}.$$

Then, we have

$$T \mathrm{diag}(\mathcal{A}, \mathcal{P}\mathcal{A}\mathcal{P}^\dagger) T^{-1} = \begin{bmatrix} \mathcal{P}\mathcal{A}\mathcal{P}^\dagger & -\mathcal{P}\mathcal{A}\overline{\mathcal{P}}^\dagger \overline{\mathcal{P}} \\ 0 & \mathcal{A} \end{bmatrix}$$

$$T \begin{bmatrix} \mathcal{B} \\ \mathcal{P}\mathcal{B} \end{bmatrix} = T \begin{bmatrix} 0 \\ \mathcal{B} \end{bmatrix}$$

and $[\mathcal{S} \ \ -\mathcal{S}\mathcal{P}^\dagger] T^{-1} = [-\mathcal{S}\mathcal{P}^\dagger \ \ \mathcal{S}\overline{\mathcal{P}}^\dagger \overline{\mathcal{P}}]$, where

$$\overline{\mathcal{P}} := \left[\overline{W} V_{\frac{1}{2}}, 0\right], \quad \overline{\mathcal{P}}^\dagger := \left[\overline{W} V_{\frac{1}{2}}^{-1}, 0\right]^\mathsf{T}$$

for $\overline{W} \in \mathbb{R}^{(n-\hat{n})\times n}$ such that $[W^\mathsf{T}, \overline{W}^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{n\times n}$ is unitary. Note that

$$\overline{\mathcal{P}}^\dagger \overline{\mathcal{P}} = \mathcal{J}\overline{\mathcal{P}}^\dagger \overline{\mathcal{P}}, \quad \mathcal{J} := \mathrm{diag}(I_n, 0)$$

holds from the block structure of $\overline{\mathcal{P}}^\dagger \overline{\mathcal{P}}$. Thus, it follows that

$$\|e_z - \hat{e}_z\|_{\mathcal{L}_2} = \|\Theta(s)\mathcal{J}\overline{\mathcal{P}}^\dagger \overline{\mathcal{P}}(sI_{2n} - \mathcal{A})^{-1}\mathcal{Z}\|_{\mathcal{H}_2}$$

where

$$\Theta(s) := \mathcal{S}\mathcal{P}^\dagger(sI_{n+\hat{n}} - \mathcal{P}\mathcal{A}\mathcal{P}^\dagger)^{-1}\mathcal{P}\mathcal{A} + \mathcal{S}$$

and $\mathcal{Z}$ satisfies

$$\mathcal{Z}\mathcal{Z}^\mathsf{T} = \mathcal{B}\mathcal{B}^\mathsf{T} + \mathcal{X}_0\mathcal{X}_0^\mathsf{T}.$$

Since $SV_{\frac{1}{2}}^{-1}\overline{W}^\mathsf{T} = 0$ follows from the first condition in (16), we can verify that

$$\Theta(s)\mathcal{J}\overline{\mathcal{P}}^\dagger \overline{\mathcal{P}} = [\theta(s)V_{\frac{1}{2}}^{-1}, 0]\mathrm{diag}(\overline{W}^\mathsf{T}\overline{W}V_{\frac{1}{2}}, 0)$$

holds, where

$$\theta(s) := SP^\dagger(sI_{\hat{n}} - P(A-HC)P^\dagger)^{-1}P(A-HC).$$

Thus, we obtain

$$\|e_z - \hat{e}_z\|_{\mathcal{L}_2} \leq \\ \|\theta(s)V_{\frac{1}{2}}^{-1}\|_{\mathcal{H}_\infty}\|[\overline{W}V_{\frac{1}{2}}, 0](sI_{2n} - \mathcal{A})^{-1}\mathcal{Z})\|_{\mathcal{H}_2}.$$

Lemma 1 with (15) guarantees $\|\theta(s)V_{\frac{1}{2}}^{-1}\|_{\mathcal{H}_\infty} < \gamma$. Furthermore, from simple calculation, we verify that

$$\mathcal{Z}\mathcal{Z}^\mathsf{T} \preceq \mathcal{B}\mathcal{B}^\mathsf{T} + \alpha^2\mathrm{diag}(0, I_n).$$

Thus, owing to the monotonicity of the solution of the Lyapunov equation, the second condition in (16), which can be rewritten as

$$\sqrt{\mathrm{tr}(\overline{W}\Phi\overline{W}^\mathsf{T})} = \sqrt{\mathrm{tr}(\overline{W}V_{\frac{1}{2}}\mathcal{K}_{1:n}V_{\frac{1}{2}}^\mathsf{T}\overline{W}^\mathsf{T})} \leq \epsilon.$$

This guarantees that

$$\|[\overline{W}V_{\frac{1}{2}}, 0](sI_{2n} - \mathcal{A})^{-1}\mathcal{Z})\|_{\mathcal{H}_2} \leq \epsilon.$$

Hence, the claim follows. ∎

Theorem 1 provides an appropriate biorthogonal projection to approximate the Luenberger-type functional observer Σ_{obs} in (3). This result is novel in the sense that, while explicitly taking into account the dynamics of Σ in (1) and Σ_{obs} in (3), the performance degradation is evaluated in terms of the $\mathcal{L}_2$-norm as shown in (18). Note that ϵ in (18) can be used as a design criterion to regulate the approximating quality of the resultant low-dimensional observer.

Remark 1: To find $W \in \mathbb{R}^{\hat{n}\times n}$ such as $WW^\mathsf{T} = I_{\hat{n}}$ and (16) for a prescribed ϵ, we can use the following procedure: First, we find the set $\{(\lambda_i, v_i)\}_{i\in\{1,\dots,n\}}$ of all eigenpairs of Φ in (17), assuming without loss of generality $\lambda_i \geq \lambda_{i+1}$ and $\|v_i\| = 1$. Next, we find $m \in \{1,\dots,n\}$ such that

$$\epsilon^2 \geq \lambda_{m+1} + \cdots + \lambda_n \tag{19}$$

and construct $V_m = [v_1, \dots, v_m] \in \mathbb{R}^{n\times m}$. Finally, by the Gram-Schmidt process, we derive W such that

$$\mathrm{im}(W^\mathsf{T}) = \mathrm{im}([V_m, (SV_{\frac{1}{2}}^{-1})^\mathsf{T}])$$

holds.

Remark 2: Even though the value of $\|x_0\|$ is not available when we construct low-dimensional observers, α in (14) can be used as a design parameter to assign a proportion of the approximating quality between the input response and initial value response. For example, if we give a larger α, then a lager initial value x_0 can be allowed, while a lager performance degradation is possibly caused by the effect of an input signal u, and vice versa.

B. Design Procedure of Low-Dimensional Functional Observer

In this subsection, we describe a procedure for constructing a low-dimensional observer that admits a prescribed $\mathcal{L}_2$-error bound. First, for a pair (V, γ) satisfying (15), we notice that (cV, γ) also satisfies (15) for any $c \geq 1$. On the other hand, since the eigenvalues of Φ in (17) linearly increase with the scale of V, a smaller V is desirable for the minimization of ϵ in (16). In view of this, it is reasonable to find a pair (V, γ) such that (15) while minimizing β for

$$V \prec \beta I_n. \tag{20}$$

In conjunction with this additional minimization, a procedure to solve Problem 1 is provided as follows:

1) For a given stable system Σ in (1), design $H \in \mathbb{R}^{n\times m_y}$ in (3) that realizes a desired trajectory of the observation error $e_z \in \mathbb{R}^{n\times m_z}$ in (4).
2) Find a pair (V, γ) such that (15) while minimizing $\beta > 0$ for (20).
3) For fixed constants $\alpha \geq 0$ and $\epsilon \geq 0$, find $W \in \mathbb{R}^{\hat{n}\times n}$ such that $WW^\mathsf{T} = I_{\hat{n}}$ and (16), with the procedure given in Remark 1.
4) Compute the biorthogonal projection as $P = WV_{\frac{1}{2}}$ and $P^\dagger = V_{\frac{1}{2}}^{-1}W^\mathsf{T}$.
5) Construct the low-dimensional functional observer $\hat{\Sigma}_{\mathrm{obs}}$ in (2) with (5).

The efficiency of this design procedure is demonstrated through a numerical example in the following section.

IV. Numerical Example

A. Power Network Model

In this section, we deal with the following electric power network system that consists of generators and loads [13], [14]. Let N denote the number of generators. For each $i \in \{1,\dots,N\}$, the dynamics of the i-th generator is modeled as

$$\Sigma_i^g : \begin{cases} \dot{\phi}_i = \tilde{A}_i\phi_i + \frac{1}{M_i}\tilde{b}p_i^g \\ \delta_i^g = \tilde{c}\phi_i \end{cases} \tag{21}$$

where $\phi_i \in \mathbb{R}^4$ denotes the states of a prime mover and a governor respectively, $p_i^g \in \mathbb{R}$ denotes the output power of a generator, $\delta_i^g \in \mathbb{R}$ denotes the electric angle of a generator, and

$$\tilde{A}_i := \begin{bmatrix} A_i^f & \frac{-1}{M_i} bc \\ k_i b b^\mathsf{T} & A_i^c \end{bmatrix},\ \tilde{b} := \begin{bmatrix} b \\ 0 \end{bmatrix},\ \tilde{c} := \begin{bmatrix} c & 0 \end{bmatrix}$$

for

$$\begin{aligned} A_i^f &:= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{D_i}{M_i} \end{bmatrix}, \quad A_i^c := \begin{bmatrix} -\frac{1}{T_i} & \frac{1}{T_i} \\ 0 & -k_i R_i \end{bmatrix} \\ b &:= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad\qquad c := \begin{bmatrix} 1 & 0 \end{bmatrix}. \end{aligned}$$

Note that the generator Σ_i^g in (21) consists of a negative feedback interconnection of a governor and a prime mover, whose system matrices are described by (A_i^f, b, c) and (A_i^c, b, c), respectively.

Furthermore, we suppose that a load is modeled as rotational mass damper system for simplicity. Let L denote the number of loads. Then, for each $i \in \{1, \ldots, L\}$, the dynamics of the i-th load is represented as

$$\Sigma_i^l : \begin{cases} \dot{\psi}_i = A_i^f \psi_i + \frac{1}{M_i} b p_i^l + \kappa_i b u_i \\ \delta_i^l = c\psi_i \end{cases} \tag{22}$$

where $\psi_i \in \mathbb{R}^2$ denotes the angle and the angular velocity of a rotor respectively, $p_i^l \in \mathbb{R}$ denotes the output power of a load, $u_i \in \mathbb{R}$ denotes the power consumption of a load, and $\kappa_i \in \mathbb{R}$ denotes a sensitivity gain. By letting

$$\begin{aligned} p &:= [p_1^g, \ldots, p_N^g, p_1^l, \ldots, p_L^l]^\mathsf{T} \\ \delta &:= [\delta_1^g, \ldots, \delta_N^g, \delta_1^l, \ldots, \delta_L^l]^\mathsf{T}, \end{aligned}$$

the interconnection among the generators and loads can be described by

$$p = -Y\delta \tag{23}$$

where $Y \in \mathbb{R}^{(N+L)\times(N+L)}$ denotes an admittance matrix. In addition, we take the measurement output y and the signal z in (1) as

$$\begin{aligned} y &:= [\phi_1^\mathsf{T}, \ldots, \phi_N^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{4N} \\ z &:= \mathrm{diag}(b^\mathsf{T}, \ldots, b^\mathsf{T})[\psi_1^\mathsf{T}, \ldots, \psi_L^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^L. \end{aligned}$$

This means that y is taken as the states of all generators and z is taken as the angular velocity of all loads.

In this model, since Y is given by a Graph Laplacian, the generators and loads are interacted by the difference of δ. It means that any bias of δ does not affect the interaction. Since the model has a zero eigenvalue, we establish a reference value of δ. To this end, we assume, without loss of generality, that the first generator is a reference component. According to this, we define the state variable as

$$x := [(r_1^g)^\mathsf{T}, \ldots, (r_N^g)^\mathsf{T}, (r_1^l)^\mathsf{T}, \ldots, (r_L^l)^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^n \tag{24}$$

where $n := 4N + 2L - 1$ and

$$\begin{aligned} r_i^g &:= \begin{cases} [0, I_3]\phi_1, & i = 1 \\ \phi_i - \tilde{c}^\mathsf{T}\tilde{c}\phi_1, & i \in \{2, \ldots, N\} \end{cases} \\ r_i^l &:= \psi_i - c^\mathsf{T} c\phi_1, \quad i \in \{1, \ldots, L\}. \end{aligned}$$

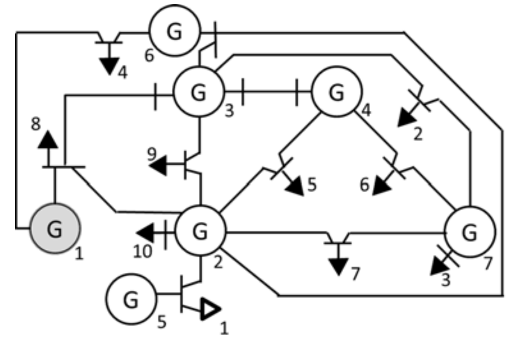

Fig. 2. Electric power network system. Circles and triangles denote generators and loads respectively. Reference generator is denoted as filled circle.

Note that, by using $\mathbf{1}_n := [1, \ldots, 1]^\mathsf{T} \in \mathbb{R}^n$ and the Kronecker product $\otimes$, (24) can be rewritten by

$$x = \mathcal{T}[\phi_1^\mathsf{T}, \ldots . \phi_N^\mathsf{T}, \psi_1^\mathsf{T}, \ldots, \psi_L^\mathsf{T}]^\mathsf{T}$$

where $\mathcal{T} := [0, I_n]\, T \in \mathbb{R}^{n\times(n+1)}$ and

$$T := \begin{bmatrix} I_4 & 0 \\ -\begin{bmatrix} \mathbf{1}_{N-1} \otimes \tilde{c}^\mathsf{T}\tilde{c} \\ \mathbf{1}_L \otimes c^\mathsf{T} c \end{bmatrix} & I_{n-3} \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}.$$

Consequently, we can represent the dynamics of the power network as Σ in (1) with

$$\begin{aligned} A &= \mathcal{T}\{\mathrm{diag}(\tilde{A}_1, \ldots, \tilde{A}_N, A_1^f, \ldots A_L^f) - \tilde{B} Y \tilde{C}\}\mathcal{T}^\dagger \\ B &= \mathcal{T} \begin{bmatrix} 0_{(n+1-2L)\times L} \\ \mathrm{diag}(\kappa_1 b, \ldots, \kappa_L b) \end{bmatrix} \\ C &= \begin{bmatrix} I_{4N} & 0_{4N\times(n+1-4N)} \end{bmatrix} \mathcal{T}^\dagger \\ D &= 0_{4N\times L} \\ S &= \begin{bmatrix} 0_{L\times(n+1-2L)} & \mathrm{diag}(b^\mathsf{T}, \ldots, b^\mathsf{T}) \end{bmatrix} \mathcal{T}^\dagger \end{aligned}$$

where $0_{m\times n}$ denotes the zero matrix in $\mathbb{R}^{m\times n}$, $\mathcal{T}^\dagger := T^{-1}[0, I_n]^\mathsf{T} \in \mathbb{R}^{(n+1)\times n}$ and

$$\begin{aligned} \tilde{B} &:= \mathrm{diag}\left(\tfrac{1}{M_1}, \ldots, \tfrac{1}{M_N}, \tfrac{1}{M_1}, \ldots, \tfrac{1}{M_L}\right) \mathrm{diag}(I_N \otimes \tilde{b}, I_L \otimes b) \\ \tilde{C} &:= \mathrm{diag}(I_N \otimes \tilde{c}, I_L \otimes c). \end{aligned}$$

B. Design of Low-dimensional Functional Observer

In this subsection, we demonstrate the efficiency of our method to construct low-dimensional observers. In what follows, we use an electric power network composed of $N = 7$ generators and $L = 10$ loads, which lead to a 47-dimensional system. Furthermore, the parameters of the system are fixed as follows: $k_i = 10$ for all i, the values of R_i, D_i, T_i and M_i are randomly chosen from $\{0.05, 0.1\}$, $\{2.5, 3.5, 5.5\}$, $\{1, 4\}$ and $\{1.6, 5\}$, respectively, and the nonzero elements of Y are chosen from $[0.1, 1]$. A depiction of this system is given in Fig. 2, which shows the interconnection structure of the generators and loads.

A Luenberger-type functional observer Σ_{obs} in (3) is designed based on pole placement. By letting $\gamma = 5.0$, we find a solution V such that (15) while minimizing β for

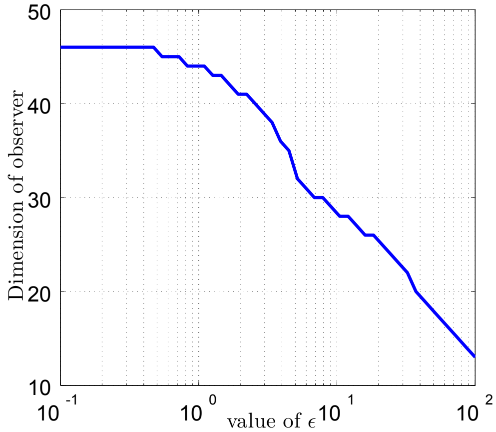

Fig. 3. Order of Resulting Models versus Values of ϵ.

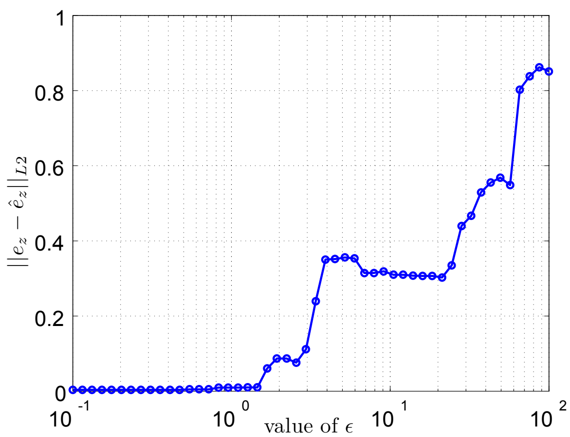

Fig. 4. Performance degradation versus Values of ϵ.

(20). In Fig. 3, we plot the resultant dimension of reduced observers versus the values of ϵ. Furthermore, we plot in Fig. 4 the resultant performance degradation $\|e_z - \hat{e}_z\|_{\mathcal{L}_2}$ in (7) versus the values of ϵ. From these figures, we can see that the parameter ϵ works a design criterion that regulates the approximating quality of the resultant low-dimensional observer.

Finally, Fig. 5 shows the trajectories of z in (1), $z_{\rm obs}$ in (3), and $\hat{z}_{\rm obs}$ in (2). In this figure, we only show the trajectories of the first elements of z, $z_{\rm obs}$ and $\hat{z}_{\rm obs}$, since the others behave similarly to the first elements. We can see from this figure that the trajectories of $\hat{z}_{\rm obs}$ appropriately approach to that of $z_{\rm obs}$ as increasing the dimension $\hat{n}$, which is a decreasing function of ϵ as shown in Fig. 3. In addition, both trajectories of $\hat{z}_{\rm obs}$ converge to that of z within 10 [sec], though the state of the original system Σ does not still converge to zero.

V. Conclusion

In this paper, we have proposed a design method for low-dimensional linear functional observers by taking a model reduction approach. The proposed method guarantees stability preservation and an a priori $\mathcal{L}_2$-error bound for the observer approximation. The efficiency of the proposed method has been shown through an example of electric power network systems.

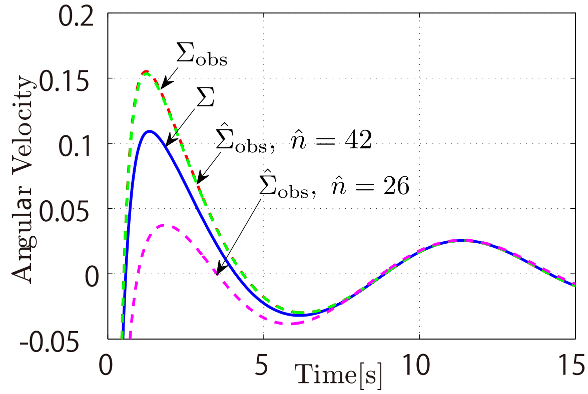

Fig. 5. Responses of angular velocity of the load shown as an outlined triangle in Fig. 2.

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