

Dissipativity-Preserving Model Reduction for Large-Scale Distributed Control Systems

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Abstract—We propose a dissipativity-preserving structured model reduction method for distributed control systems. As a fundamental tool to develop structured model reduction, we first establish dissipativity-preserving model reduction for general linear systems on the basis of a singular perturbation approximation. To this end, by deriving a tractable expression of singular perturbation models, we characterize dissipativity preservation in terms of a projection-like transformation of storage functions, and we show that the resultant approximation error is relevant to the sum of neglected eigenvalues of an index matrix. Next, utilizing this dissipativity-preserving model reduction, we develop a structured controller reduction method for distributed control systems. The major significance of this method is to preserve the spatial distribution of dissipative controllers and to provide an a priori bound for the performance degradation of closed-loop systems in terms of the \mathcal{H}_2 -norm. The efficiency of the proposed method is verified through a numerical example of vibration suppression control for interconnected second-order systems.

Index Terms—Structured Model Reduction; Dissipativity Preservation; Distributed Control Systems; Singular Perturbation Approximation.

I. INTRODUCTION

RECENT developments in computer networking technology have enabled us to analyze and synthesize control systems in a spatially distributed manner. Such distributed control system designs have good compatibility with the spatial distribution of physical plants typically found in power systems, building thermal systems, industrial processes, and so forth; see [1]–[3] and references therein. In particular, over the past several years, networked control design for cyber-physical systems has attracted attention from academia as well as from industry [4], [5]. Against this background, it can be widely expected that the demands on the distributed analysis and synthesis of physical systems will increase.

However, because many physical systems can be modeled as large-scale (i.e., high-dimensional) dynamical systems, this naturally makes the architecture of the associated controllers

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more complex and larger in scale. In view of this, it is desirable that the architecture of the controllers to be simplified while guaranteeing the performance of control systems. From a systems-theory perspective, such a problem can be formulated as a model reduction problem for controllers, which is called a controller reduction problem.

A number of controller reduction methods can be found in the literature; see [6]–[9] and references therein. However, even though their efficiency has been intensively investigated, the application of existing controller reduction methods to control systems having a spatial distribution poses a challenge. That is, the reduced controllers obtained by standard methods do not conform to the physical restrictions imposed by the environment, such as limitations of sensor and actuator allocations. This is because the standard methods do not consider the spatial distribution of the controllers. In this sense, a novel *structured* controller reduction method is indispensable to controller reduction for distributed control systems to comply with physical restrictions. Note that such a structured problem is much more challenging than the standard reduction problem for models and controllers.

To address this difficult problem in a scalable manner, we confine the class of systems to one that possesses a *physical* property, focusing on a class of systems that dissipate some of the physical energy (or perhaps virtual energy). Such energy dissipation is mathematically formulated as system *passivity*, which is often used for designing control systems to guarantee closed-loop stability [10]–[16]. The main contribution of this paper is to develop structured controller reduction for distributed control systems by fully taking advantage of a passivity-based analysis. Our approach is summarized as follows:

- (i) We first develop a passivity-preserving model reduction method for general linear systems on the basis of a singular perturbation approximation. The major result consists in not only deriving a condition for passivity preservation but also developing a novel \mathcal{H}_2 -error analysis of the approximation. This serves as a fundamental tool for solving the structured controller reduction problem.
- (ii) We investigate the condition under which the appropriate structured controller reduction is achieved by directly applying the passivity-preserving model reduction to closed-loop systems, thereby developing a structured controller reduction method for distributed control systems.

It should be noted that our approach is entirely different from that of the existing methods [6]–[9], most of which use a

standard \mathcal{H}_∞ -control framework, having good compatibility with robust stability and performance analyses based on the graph, gap and ν -metrics [17]–[19]. This novel approach enables us to not only robustly guarantee the stability of reduced closed-loop systems regardless of the amplitude of controller reduction errors, but also achieve the preservation of the spatial distribution of controllers.

To clarify our contribution, some references for structure-preserving model reduction and the singular perturbation approximation are in order. As for structure-preserving model reduction, [20] and [21] each address a model reduction problem while preserving a particular system structure such as the Lagrangian structure or the second-order structure. However, they are formulated neither on the premise of controller reduction nor passivity preservation. On the other hand, even though [22], [23] and [24] develop model reduction methods with passivity preservation, they can be applied to only disjoint subsystems or controllers. Thus, no global error bound is provided for the approximation of interconnected systems.

As examples of one approach similar to ours, there are singular perturbation methods based on state aggregation [25], [26]. However, such a line of inquiry does not explicitly take into account the effect of external inputs. By considering input and output mappings, a structure-preserving singular perturbation approximation is developed in [27], where an error expression in terms of the Hankel norm is derived for the error systems with a specific structure imposed on the initial values. Even though the specific structure of the initial values has good compatibility with the singular perturbation approximation, the quality of approximation rests potentially on a priori system decomposition into subsystems with different time scales. It is known that the systematic implementation of such decomposition possibly becomes an issue especially for large-scale systems.

It should be further noted that our \mathcal{H}_2 -error analysis is different from that in [28]–[30], which use asymptotic analyses in the time domain. In contrast to this, we analyze the approximation error in the Laplace domain by deriving a novel representation for the error systems, which leads to clear insight into regulating the approximation quality of resultant approximate models.

In addition, a state aggregation method based on network clustering has been developed for network structure-preserving model reduction [31]–[33]. In this method, we find a set of states that behave similarly for input signals, called clusters, and then, we use a block-diagonally structured projection to construct an approximate model that preserves the interconnection topology among the clusters. By incorporating the network structure preservation into a singular perturbation approximation, a structured controller reduction method is developed in this paper. Furthermore, the \mathcal{H}_2 -error analysis is based on the factorization of the transfer matrix of error systems, which corresponds to a counterpart for the state aggregation with orthogonal projection. Finally, we provide detailed proofs omitted in the preliminary version [34], and conduct additional numerical experiments to compare the performance of our method with that of existing model and controller reduction methods.

This paper is organized as follows. In Section II, we first formulate the problem of the structured controller reduction while explaining its difficulty. In Section III, as a fundamental tool to give a solution to the problem, we develop a dissipativity-preserving model reduction method for general linear systems on the basis of a singular perturbation approximation. It should be noted that the formulation of dissipativity includes that of passivity as a special case. The major result here consists in not only deriving a condition for dissipativity preservation but also developing a novel \mathcal{H}_2 -error analysis of the singular perturbation approximation in the Laplace domain. In Section IV, we give a solution to the structured controller reduction problem for distributed control systems utilizing the dissipativity-preserving model reduction. Furthermore, we provide an algorithm to systematically implement the structured controller reduction and demonstrate the efficiency of the proposed algorithm through a numerical example, where passive controller reduction for vibration suppression control is considered. Finally, concluding remarks are provided in Section V.

Notation. The following notation is used in this paper:

\mathbb{R}	set of real numbers
I_n	n -dimensional identity matrix
$\text{tr}(M)$	trace of a matrix M
$\text{im}(M)$	image of a matrix M
$\text{rank}(M)$	rank of a matrix M
$M \succ \mathcal{O}_n$ ($M \prec \mathcal{O}_n$)	positive (negative) definiteness of a symmetric matrix $M \in \mathbb{R}^{n \times n}$
$M \succeq \mathcal{O}_n$ ($M \preceq \mathcal{O}_n$)	positive (negative) semidefiniteness of a symmetric matrix $M \in \mathbb{R}^{n \times n}$

The block diagonal matrix having matrices M_1, \dots, M_n on its block diagonal is denoted by

$$\text{dg}(M_1, \dots, M_n) = \text{dg}(M_i)_{i \in \{1, \dots, n\}}.$$

The \mathcal{H}_∞ -norm of a stable proper transfer matrix G and the \mathcal{H}_2 -norm of a stable strictly proper transfer matrix G are respectively defined by

$$\begin{aligned} \|G(s)\|_{\mathcal{H}_\infty} &:= \sup_{\omega \in \mathbb{R}} \sigma(G(j\omega)), \\ \|G(s)\|_{\mathcal{H}_2} &:= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}(G(j\omega)G^T(-j\omega)) d\omega \right)^{\frac{1}{2}} \end{aligned}$$

where $\sigma(\cdot)$ denotes the maximum singular value.

II. PROBLEM FORMULATION

In this section, we formulate a structured controller reduction problem for large-scale physical systems. Figure 1 depicts a control system for the case where a controlled plant and several controllers communicate through sensing and actuation. In this system, the set of controllers is distributed over a plant in compliance with some physical restrictions. Examples of such a distributed control system include building thermal systems, in which the temperature of each room is regulated by an air-conditioning system driven on the basis of local sensor information.

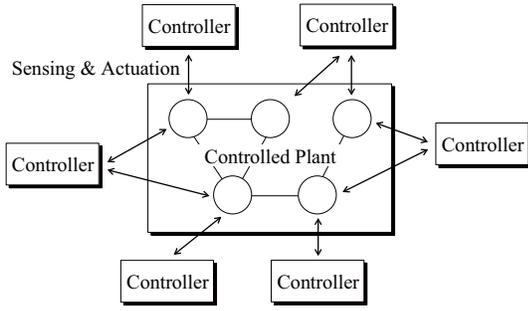


Fig. 1. Schematic depiction of distributed control systems.

In this paper, we describe the dynamics of the distributed control system by

$$\begin{aligned} \Sigma_0 : \begin{cases} \dot{x}_0 = A_0 x_0 + B_0 u + \sum_{l=1}^L b_{0,l} w_l \\ y = C_0 x_0 + D_0 u \\ z_l = c_{0,l} x_0, \end{cases} \\ \Sigma_l : \begin{cases} \dot{x}_l = A_l x_l + b_l z_l \\ w_l = c_l x_l + d_l z_l, \end{cases} \quad l \in \mathbb{L} \end{aligned} \quad (1)$$

where Σ_0 and Σ_l for $l \in \mathbb{L} := \{1, \dots, L\}$ denote a plant and a set of controllers, u and y denote a control input and an evaluated output, and z_l and w_l denote the sensor and actuator signal associated with Σ_l . The signal communication structure of this closed-loop system is depicted to the left in Fig. 2, where $z := [z_1^\top, \dots, z_L^\top]^\top$ and $w := [w_1^\top, \dots, w_L^\top]^\top$. In the following, we denote the distributed control system by $(\Sigma_0, \{\Sigma_l\}_{l \in \mathbb{L}})$.

With recent technical developments, the scale of systems of interest to the control community has tended to become larger, and this has naturally made the architecture of the associated controllers more complex. In view of this, it is desirable that the architecture of controllers be simplified while keeping the performance of closed-loop systems. To formulate a controller reduction problem, we describe the approximate model of the distributed control system by

$$\begin{aligned} \Sigma_0 : \begin{cases} \dot{\hat{x}}_0 = A_0 \hat{x}_0 + B_0 u + \sum_{l=1}^L b_{0,l} \hat{w}_l \\ \hat{y} = C_0 \hat{x}_0 + D_0 u \\ \hat{z}_l = c_{0,l} \hat{x}_0, \end{cases} \\ \hat{\Sigma}_l : \begin{cases} \dot{\hat{x}}_l = \hat{A}_l \hat{x}_l + \hat{b}_l \hat{z}_l \\ \hat{w}_l = \hat{c}_l \hat{x}_l + \hat{d}_l \hat{z}_l, \end{cases} \quad l \in \mathbb{L}, \end{aligned} \quad (2)$$

where each original controller Σ_l is replaced with its approximate model $\hat{\Sigma}_l$. We replace the state variable of Σ_0 with \hat{x}_0 , because the trajectories of x_0 and \hat{x}_0 are made different by the approximation error of the controllers. The structure of this approximate model is depicted to the right in Fig. 2, where $\hat{z} := [\hat{z}_1^\top, \dots, \hat{z}_L^\top]^\top$ and $\hat{w} := [\hat{w}_1^\top, \dots, \hat{w}_L^\top]^\top$. In the following, we denote the approximate model by $(\Sigma_0, \{\hat{\Sigma}_l\}_{l \in \mathbb{L}})$. It can easily be verified that the spatial distribution of the controllers is preserved by this formulation. This paper is aimed at finding a set of approximate controllers $\hat{\Sigma}_l$ such that the discrepancy between y and \hat{y} is sufficiently small in a suitable sense.

A simple method of realizing such a structured controller reduction is to apply an existing model reduction method to each controller Σ_l . However, this approach possibly leads

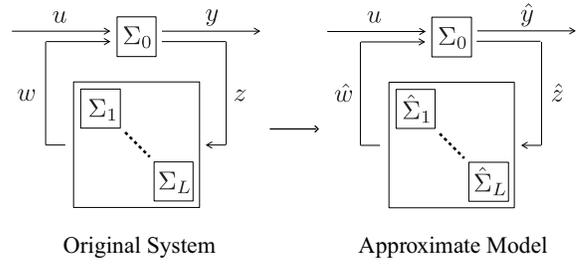


Fig. 2. Signal communication structures of original system and approximate model.

to an undesirable result, because the performance of the entire control system may be degraded by even a small error in the approximation of the controllers. More specifically, even if each controller is well approximated, the performance degradation of the closed-loop system may become large. That is, even if the dynamics from z to w in Fig. 2 is well approximated, the approximation error of the dynamics from u to y may become large.

To address this issue, we confine the class of systems to one that possesses a *physical* property, namely *passive* systems; see Section III. It is known that certain interconnections of passive components retain the passivity, thereby guaranteeing the stability of closed-loop systems. In this sense, passivity-based control design is especially efficient for control systems that are complex or large-scale. With the premise of passivity preservation, we address the following structured controller reduction problem for distributed control systems.

Problem: Let a distributed control system $(\Sigma_0, \{\Sigma_l\}_{l \in \mathbb{L}})$ in (1) be given, and assume that the plant Σ_0 and each controller Σ_l are passive. Find an approximate model $(\Sigma_0, \{\hat{\Sigma}_l\}_{l \in \mathbb{L}})$ in (2) such that each approximate controller $\hat{\Sigma}_l$ remains passive and the discrepancy between y and \hat{y} is small enough in a suitable sense.

In the formulation above, to simplify the arguments, communication among controllers is not introduced in (1), with similar results available also in the case where some communication is allowed. More specifically, we can introduce communication among controllers to (1) as

$$\Sigma_l : \begin{cases} \dot{x}_l = A_l x_l + b_l z_l + f_l \sum_{k=1}^L \gamma_{l,k} v_k \\ w_l = c_l x_l + d_l z_l \\ v_l = g_l x_l, \end{cases} \quad (3)$$

where v_l denotes a communication signal among controllers, and $\Gamma := \{\gamma_{i,j}\}$ represents an interconnection structure of controllers.

III. DISSIPATIVITY-PRESERVING SINGULAR PERTURBATION APPROXIMATION

A. Mathematical Formulation

In this subsection, we mathematically formulate a model reduction framework based on a singular perturbation approximation. Let us consider a linear system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad (4)$$

$$\mathcal{F}_Q(A, B, C, D; V) := \begin{bmatrix} A^T V + V A & V B \\ B^T V & 0 \end{bmatrix} - \begin{bmatrix} C^T & 0 \\ D^T & I_m \end{bmatrix} Q \begin{bmatrix} C & D \\ 0 & I_m \end{bmatrix} \quad (15)$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{q \times n}$ and $D \in \mathbb{R}^{q \times m}$. In much of the literature on singular perturbation theory [28]–[30], a time scale separation of Σ in (4) is commonly used to motivate a singular perturbation approximation, while it is not always assumed as exemplified by the balanced residualization [35]–[37]. Such a time scale separation is not used in this paper. Instead, by finding an appropriate coordinate transformation, we decouple Σ into two subsystems in a general manner. Note that [25], [26] consider a singular perturbation approximation in a setting based on state aggregation. However, such a line of inquiry does not explicitly take into account the effect of external inputs.

In the following, we denote the set of projection matrices by

$$\mathcal{P}^{\hat{n} \times n} := \{P \in \mathbb{R}^{\hat{n} \times n} : P P^T = I_{\hat{n}}, \quad \hat{n} \leq n\}, \quad (5)$$

and we perform the coordinate transformation of Σ with a unitary matrix $[P^T, \bar{P}^T]^T \in \mathbb{R}^{n \times n}$ with $P \in \mathcal{P}^{\hat{n} \times n}$ and $\bar{P} \in \mathcal{P}^{(n-\hat{n}) \times n}$. Then, we have

$$\tilde{\Sigma} : \begin{cases} \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} P A P^T & P A \bar{P}^T \\ \bar{P} A P^T & \bar{P} A \bar{P}^T \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} P B \\ \bar{P} B \end{bmatrix} u \\ y = \begin{bmatrix} C P^T & C \bar{P}^T \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + D u. \end{cases} \quad (6)$$

To reduce the dimension of $\tilde{\Sigma}$, we impose an *algebraic constraint* on the trajectory of η . More specifically, by replacing $\dot{\eta}$ in (6) with zero, we obtain

$$0 = \bar{P} A P^T \hat{\xi} + \bar{P} A \bar{P}^T \hat{\eta} + \bar{P} B u \quad (7)$$

where η and ξ are replaced with their approximants $\hat{\eta}$ and $\hat{\xi}$, respectively. As long as $\bar{P} A \bar{P}^T$ is nonsingular, $\hat{\eta}$ in (7) is obtained as

$$\hat{\eta} = -(\bar{P} A \bar{P}^T)^{-1} \bar{P} A P^T \hat{\xi} - (\bar{P} A \bar{P}^T)^{-1} \bar{P} B u. \quad (8)$$

This approximation is reasonable if the convergence rate of η is sufficiently greater than that of ξ . Substituting (8) into the equation with respect to $\dot{\xi}$, we have the singular perturbation model

$$\hat{\Sigma} : \begin{cases} \dot{\hat{\xi}} = \hat{A} \hat{\xi} + \hat{B} u \\ \hat{y} = \hat{C} \hat{\xi} + \hat{D} u \end{cases} \quad (9)$$

where

$$\begin{aligned} \hat{A} &:= P A P^T + P A \Pi A P^T \in \mathbb{R}^{\hat{n} \times \hat{n}}, \\ \hat{B} &:= P B + P A \Pi B \in \mathbb{R}^{\hat{n} \times m}, \\ \hat{C} &:= C P^T + C \Pi A P^T \in \mathbb{R}^{q \times \hat{n}}, \\ \hat{D} &:= D + C \Pi B \in \mathbb{R}^{q \times m}, \end{aligned} \quad (10)$$

with

$$\Pi := -\bar{P}^T (\bar{P} A \bar{P}^T)^{-1} \bar{P} \in \mathbb{R}^{n \times n}. \quad (11)$$

Note that Π does not depend on the basis selected for the projection $\bar{P} \in \mathcal{P}^{(n-\hat{n}) \times n}$. This is because

$$\Pi = -\bar{P}^T H^T (H \bar{P} A \bar{P}^T H^T)^{-1} H \bar{P}$$

for any unitary matrix $H \in \mathbb{R}^{(n-\hat{n}) \times (n-\hat{n})}$. Thus, the singular perturbation model $\hat{\Sigma}$ in (9) depends only on the choice of $P \in \mathcal{P}^{\hat{n} \times n}$. In the rest of this paper, the transfer matrix of Σ is denoted by

$$G(s) := C(sI_n - A)^{-1} B + D, \quad (12)$$

and the singular perturbation approximant of G associated with $P \in \mathcal{P}^{\hat{n} \times n}$ is denoted by

$$\hat{G}(s; P) := \hat{C}(sI_{\hat{n}} - \hat{A})^{-1} \hat{B} + \hat{D}, \quad (13)$$

where \hat{A} , \hat{B} , \hat{C} and \hat{D} are defined as in (10). In the following subsections, we investigate how the selection of $P \in \mathcal{P}^{\hat{n} \times n}$ affects the property of the approximant \hat{G} .

Remark 1: The singular perturbation approximation exactly preserves the zero frequency gain of the original system for any $P \in \mathcal{P}^{\hat{n} \times n}$; see Theorem 3 below for a proof. On the other hand, it is known that the projection-based model reduction, in which the approximate model is given by the system matrices of $P A P^T$, $P B$, $C P^T$ and D , tends to cause a larger approximation error of the zero frequency gain, while the infinite frequency gain is exactly preserved. In view of this, it can be expected that in practice, the singular perturbation approximation yields a better approximation than the projection-based model reduction, because systems appearing in practical applications often possess a low-pass property rather than a high-pass property.

B. Dissipativity Preservation

In this subsection, we derive a tractable condition under which the singular perturbation approximation duly preserves the system dissipativity. Let us begin with the following standard definition of strict dissipativity [10]–[12].

Definition 1: A linear system Σ in (4) is said to be *V-dissipative with respect to* $Q = Q^T \in \mathbb{R}^{(m+q) \times (m+q)}$ if there exists $V = V^T \succ \mathcal{O}_n$ such that

$$\mathcal{F}_Q(A, B, C, D; V) \prec \mathcal{O}_{n+m} \quad (14)$$

where \mathcal{F}_Q is defined as in (15).

In linear systems theory, (14) is called a *dissipation inequality*, and the quadratic functions

$$f_V(x) := x^T V x \quad (16)$$

and

$$s_Q(y, u) := \begin{bmatrix} y^T & u^T \end{bmatrix} Q \begin{bmatrix} y \\ u \end{bmatrix} \quad (17)$$

are called *storage functions* and *supply functions*, respectively. It is known that the dissipation inequality is equivalent to

$$\dot{f}_V(x) < s_Q(y, u) = y^T Q_{y,y} y + 2y^T Q_{y,u} u + u^T Q_{u,u} u \quad (18)$$

along the trajectory of Σ in (4) for

$$Q = \begin{bmatrix} Q_{y,y} & Q_{y,u} \\ Q_{y,u}^T & Q_{u,u} \end{bmatrix}.$$

Letting $u(t) \equiv 0$ in (18) verifies that Σ is stable whenever it is V -dissipative with respect to any Q satisfying $Q_{y,y} \preceq Q_q$. In this case, f_V in (16) can be regarded as a Lyapunov function to prove its stability. In particular, a linear system Σ in (4) is said to be *passive* if it is V -dissipative with respect to

$$Q = \begin{bmatrix} 0 & I_m \\ I_q & 0 \end{bmatrix}, \quad m = q. \quad (19)$$

In this sense, the formulation of dissipativity includes that of passivity as a special case.

The following lemma, which can easily be derived from the definition of dissipativity, is useful for the arguments made below.

Lemma 1: Let a linear system Σ in (4) be given, and suppose that it is V -dissipative with respect to Q . Consider a Cholesky factor $V_{\frac{1}{2}}$ of V such that $V = V_{\frac{1}{2}}^T V_{\frac{1}{2}}$. Then

$$\mathcal{F}_Q(V_{\frac{1}{2}}AV_{\frac{1}{2}}^{-1}, V_{\frac{1}{2}}B, CV_{\frac{1}{2}}^{-1}, D; I_n) \prec \mathcal{O}_{n+m}. \quad (20)$$

Proof: It is found that $\mathcal{F}_Q(A, B, C, D; V)$ in (15) can be rewritten as

$$\tilde{V}^T \mathcal{F}_Q(V_{\frac{1}{2}}AV_{\frac{1}{2}}^{-1}, V_{\frac{1}{2}}B, CV_{\frac{1}{2}}^{-1}, D; I_n) \tilde{V}$$

where $\tilde{V} := \text{dg}(V_{\frac{1}{2}}, I_m)$. Since $V_{\frac{1}{2}}$ is nonsingular, (14) is equivalent to (20). Hence, the claim follows. ■

Lemma 1 shows that any V -dissipative system can be transformed into a system that is I_n -dissipative with respect to the same supply function (i.e., a dissipative system that admits the purely quadratic function $x^T x$ as its storage function). Therefore, without loss of generality, we can assume that any dissipative system is I_n -dissipative.

In projection-based model reduction, such a particular realization is useful for achieving dissipativity preservation. This is because, for any $P \in \mathcal{P}^{\hat{n} \times n}$, it follows that

$$\begin{aligned} \mathcal{F}_Q(PAP^T, PB, CP^T, D; I_{\hat{n}}) \\ = \tilde{P} \mathcal{F}_Q(A, B, C, D; I_n) \tilde{P}^T \prec \mathcal{O}_{\hat{n}+m} \end{aligned}$$

where $\tilde{P} := \text{dg}(P, I_m)$, whenever (14) holds for $V = I_n$. This implies that the approximate model is $I_{\hat{n}}$ -dissipative with respect to Q whenever the original system is I_n -dissipative with respect to Q .

It should be emphasized that, due to the complicated form of \hat{A} in (10), the same conclusion for the singular perturbation approximation seems nontrivial. In fact, dissipativity preservation for a singular perturbation approximation has not been well investigated so far, though some results of passivity-preserving model reduction based on interpolation are found in [23] and [24]. In view of this, we first state the following fundamental lemma, in which a novel representation of \hat{A} is derived. This representation will provide insight into achieving dissipativity preservation in the singular perturbation approximation.

Lemma 2: For any $A \in \mathbb{R}^{n \times n}$ and $P \in \mathcal{P}^{\hat{n} \times n}$, $\hat{A} \in \mathbb{R}^{\hat{n} \times \hat{n}}$ in (10) admits the representation

$$\hat{A} = (P + PA\Pi)A(P + PA\Pi)^T, \quad (21)$$

where $\Pi \in \mathbb{R}^{n \times n}$ is defined as in (11). Moreover, $P + PA\Pi \in \mathbb{R}^{\hat{n} \times n}$ has full row rank.

Proof: First, we prove that $P + PA\Pi$ has full row rank; namely

$$\text{rank}(P + PA\Pi) = \hat{n}. \quad (22)$$

We prove this by contradiction. If $\text{rank}(P + PA\Pi) < \hat{n}$ is assumed, then we obtain

$$\begin{aligned} \text{rank}((P + PA\Pi)P^T P) \\ \leq \min(\text{rank}(P + PA\Pi), \text{rank}(P^T P)) < \hat{n}. \end{aligned}$$

However, this contradicts

$$\text{rank}((P + PA\Pi)P^T P) = \text{rank}(P) = \hat{n},$$

which follows from $\Pi P^T = 0$ and $PP^T = I_{\hat{n}}$. Thus, (22) follows.

Next, we prove the claim for \hat{A} in (21). First, we show that

$$\hat{A}P = (P + PA\Pi)A. \quad (23)$$

To this end, it suffices to show that $\hat{A}P - (P + PA\Pi)A = 0$. Using the relation of

$$-\Pi A \bar{P}^T = \bar{P}^T (\bar{P} A \bar{P}^T)^{-1} \bar{P} A \bar{P}^T = \bar{P}^T, \quad (24)$$

we obtain

$$\begin{aligned} \hat{A}P - (P + PA\Pi)A &= PA\{(I_n + \Pi A)P^T P - (I_n + \Pi A)\} \\ &= PA(I_n + \Pi A)\bar{P}^T \bar{P} \\ &= P A \bar{P}^T \bar{P} - P A \bar{P}^T \bar{P} \\ &= 0. \end{aligned}$$

Hence, (23) follows. Multiplying (23) by P^T from the right side, we obtain

$$\begin{aligned} \hat{A} &= (P + PA\Pi)A P^T \\ &= (P + PA\Pi)A(P + PA\Pi)^T - (P + PA\Pi)A(PA\Pi)^T. \end{aligned}$$

Furthermore, using (24) and

$$\Pi = \Pi \bar{P}^T \bar{P}, \quad (25)$$

we obtain

$$\begin{aligned} (P + PA\Pi)A(PA\Pi)^T &= (P + PA\Pi)A(PA\Pi \bar{P}^T \bar{P})^T \\ &= \{P A \bar{P}^T \bar{P} - P A \bar{P}^T \bar{P}\}(PA\Pi)^T \\ &= 0. \end{aligned}$$

Thus, (21) follows. ■

Lemma 2 shows that \hat{A} in $\hat{\Sigma}$ admits a *projection-like* formula as in (21). In addition, we note that \hat{B} in (10) can be rewritten as

$$\hat{B} = (P + PA\Pi)B. \quad (26)$$

However, \hat{C} is not equal to $C(P + PA\Pi)^T$ in general. Considering these facts, we can successfully derive the following theorem on dissipativity preservation.

Theorem 1: Let a linear system Σ in (4) be given, and suppose that it is I_n -dissipative with respect to Q . If $P \in \mathcal{P}^{\hat{n} \times n}$ satisfies

$$\text{im}(C^T) \subseteq \text{im}(P^T), \quad (27)$$

then the singular perturbation model $\hat{\Sigma}$ in (13) is $I_{\hat{n}}$ -dissipative with respect to Q .

$$\begin{aligned}
G(s) - \hat{G}(s; P) &= -\hat{C}(sI_{\hat{n}} - \hat{A})^{-1}PA\Pi\bar{P}^T\bar{P}A(sI_n - A)^{-1}B - C\Pi\bar{P}^T\bar{P}B \\
&\quad -\hat{C}(sI_{\hat{n}} - \hat{A})^{-1}PA\Pi\bar{P}^T\bar{P}B - C\Pi\bar{P}^T\bar{P}A(sI_n - A)^{-1}B \\
&= \left[\hat{C}(sI_{\hat{n}} - \hat{A})^{-1}(P + PA\Pi) + C\Pi \right] \bar{P}^T\bar{P} [-A(sI_n - A)^{-1}B - B]
\end{aligned} \tag{32}$$

Proof: If (27) holds, it follows that $C^T P^T P = C^T$. Using

$$P^T P + \bar{P}^T \bar{P} = I_n,$$

we have $C\bar{P}^T = 0$, which implies that $C\Pi = C\Pi^T = 0$. Thus, it follows that

$$\hat{C} = CP^T = C(P + PA\Pi)^T, \quad \hat{D} = D.$$

Noting that (26) holds, we can verify with Lemma 2 that

$$\mathcal{F}_Q(\hat{A}, \hat{B}, \hat{C}, \hat{D}; I_{\hat{n}}) = \tilde{P}\mathcal{F}_Q(A, B, C, D; I_n)\tilde{P}^T$$

where $\tilde{P} := \text{dg}(P + PA\Pi, I_m)$. Its negative definiteness follows in that (14) holds for $V = I_n$ and $P + PA\Pi$ has full row rank as shown in Lemma 2. Hence, the claim follows. ■

Theorem 1 shows that, if the original system Σ is I_n -dissipative with respect to a supply function, then the singular perturbation model $\hat{\Sigma}$ is $I_{\hat{n}}$ -dissipative with respect to the same supply function as long as (27) holds. Note that condition (27) can easily be satisfied by adding the basis of $\text{im}(C^T)$ to $\text{im}(P^T)$.

Remark 2: From systems theory, it is known that system dissipativity admits a characterization in terms of a frequency domain inequality. More specifically, if Σ in (4) is V -dissipative with respect to Q , then

$$\left[G^T(-j\omega) \quad I_m \right] Q \begin{bmatrix} G(j\omega) \\ I_m \end{bmatrix} \succ \mathcal{O}_m, \quad \forall \omega \in \mathbb{R}, \tag{28}$$

where G in (12) is the transfer matrix of Σ . This frequency domain characterization is often utilized in $\mathcal{H}_2/\mathcal{H}_\infty$ -control synthesis, for example. In view of this, the dissipativity preservation in Theorem 1 can be rephrased as the preservation of a frequency property specified by (28).

Remark 3: In Definition 1, we have introduced the strict notion of dissipativity; that is, the definiteness of V and \mathcal{F}_Q in (14) is assumed to be strict. Consequently, the existence of the Cholesky factorization of V is ensured, and therefore, the dissipativity is characterized without a controllability assumption [10]–[12]. A generalization of the dissipativity preservation to the case of semidefinite V and \mathcal{F}_Q is currently under investigation.

C. Approximation Error Analysis

In this subsection, we analyze the approximation error caused by the singular perturbation approximation. In the literature on singular perturbation theory, most of the error analyses are performed in the time domain by using asymptotic analysis [25], [26], [28]–[30], or on the basis of the balanced realization [35]–[37]. By contrast, we analyze the approximation error in the Laplace domain without relying on asymptotic analysis or the balanced realization. To this end,

a novel representation for the error system is derived in the following theorem.

Theorem 2: Let a transfer matrix G in (12) be given, and define the singular perturbation approximant \hat{G} in (13) associated with $P \in \mathcal{P}^{\hat{n} \times n}$. Then

$$G(s) - \hat{G}(s; P) = \hat{\Xi}(s; P)\bar{P}^T\bar{P}X(s) \tag{29}$$

where

$$\begin{aligned}
\hat{\Xi}(s; P) &:= \hat{C}(sI_{\hat{n}} - \hat{A})^{-1}(P + PA\Pi) + C\Pi, \\
X(s) &:= -A(sI_n - A)^{-1}B - B,
\end{aligned} \tag{30}$$

with \hat{A} and \hat{C} defined as in (10).

Proof: Denote the error system by

$$G(s) - \hat{G}(s; P) = C_e(sI_{n+\hat{n}} - A_e)^{-1}B_e + D_e$$

where $A_e = \text{dg}(\hat{A}, A)$, $B_e = [\hat{B}^T, B^T]^T$, $C_e = [-\hat{C}, C]$ and $D_e = -\hat{D} + D$. Considering the similarity transformation of the error system with

$$T = \begin{bmatrix} I_{\hat{n}} & -P \\ 0 & I_n \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} I_{\hat{n}} & P \\ 0 & I_n \end{bmatrix},$$

we have

$$\begin{aligned}
TA_eT^{-1} &= \begin{bmatrix} \hat{A} & \hat{A}P - PA \\ 0 & A \end{bmatrix}, \quad TB_e = \begin{bmatrix} PA\Pi\bar{P}^T\bar{P}B \\ B \end{bmatrix} \\
C_eT^{-1} &= \begin{bmatrix} -\hat{C} & -\hat{C}P + C \end{bmatrix}, \quad D_e = -C\Pi B,
\end{aligned} \tag{31}$$

where $I_n - P^T P = \bar{P}^T \bar{P}$ has been invoked. Using (23) and (25), we have

$$\hat{A}P - PA = PA\Pi\bar{P}^T\bar{P}A.$$

Furthermore, using (24), we obtain

$$\begin{aligned}
-\hat{C}P + C &= C(I_n - P^T P) - C\Pi A P^T P \\
&= C\bar{P}^T\bar{P} - C\Pi A(I_n - \bar{P}^T\bar{P}) \\
&= C\bar{P}^T\bar{P} - C\Pi A + C\Pi A\bar{P}^T\bar{P} \\
&= C\bar{P}^T\bar{P} - C\Pi A - C\bar{P}^T\bar{P} \\
&= -C\Pi A.
\end{aligned}$$

Thus, the block structure of (31) implies that the error system is given by (32), which proves (29). ■

The factorization of the error system in Theorem 2, which can be applied even to unstable systems, provides a qualitative insight into the error analysis. That is, from the cascaded form of (29), we expect that the resultant approximation error will be small if the norm of $\bar{P}^T\bar{P}X$ is sufficiently small, and the norm of $\hat{\Xi}$ is bounded. Note that X in (30) coincides with the transfer matrix from u to $-\dot{x}$ for the original Σ and an appropriate selection of $P \in \mathcal{P}^{\hat{n} \times n}$ can regulate the norm of $\bar{P}^T\bar{P}X$. Conversely, the norm of $\hat{\Xi}$ in (30) cannot

be arbitrarily small because it coincides with the singular perturbation approximant of

$$\Xi(s) = C(sI_n - A)^{-1}$$

associated with P . This corresponds to the state-to-output mapping of the original Σ .

Now, we are ready to state the main result of this section. By utilizing Theorem 2 in conjunction with Theorem 1, we establish the following theorem relevant to dissipativity-preserving model reduction that admits an a priori \mathcal{H}_2 -error bound.

Theorem 3: Let a linear system Σ in (4) be given, and suppose that it is I_n -dissipative with respect to Q satisfying $Q_{y,y} \preceq \mathcal{O}_q$ for (17). Let $\gamma > 0$ such that

$$A + A^\top + \gamma^{-1}(I_n + C^\top C) \prec \mathcal{O}_n. \quad (33)$$

Furthermore, let the controllability gramian $\mathcal{W} = \mathcal{W}^\top \succeq \mathcal{O}_n$ such that

$$A\mathcal{W} + \mathcal{W}A^\top + B B^\top = 0. \quad (34)$$

If $P \in \mathcal{P}^{\hat{n} \times n}$ satisfies

$$\text{im}([B, C^\top]) \subseteq \text{im}(P^\top), \quad \text{tr}(\Phi) - \text{tr}(P\Phi P^\top) \leq \epsilon^2 \quad (35)$$

where

$$\Phi := A\mathcal{W}A^\top \in \mathbb{R}^{n \times n}, \quad (36)$$

then the singular perturbation model $\hat{\Sigma}$ in (9) is $I_{\hat{n}}$ -dissipative with respect to Q and satisfies

$$G(0) = \hat{G}(0; P), \quad \|G(s) - \hat{G}(s; P)\|_{\mathcal{H}_2} \leq \gamma\epsilon \quad (37)$$

where G and \hat{G} are defined as in (12) and (13), respectively.

Proof: Owing to (35), if Σ is I_n -dissipative with respect to Q , then $\hat{\Sigma}$ is $I_{\hat{n}}$ -dissipative with respect to Q , as shown in Theorem 1. Note that both Σ and $\hat{\Sigma}$ are stable because they are I_n -dissipative and $I_{\hat{n}}$ -dissipative with respect to Q satisfying $Q_{y,y} \preceq \mathcal{O}_q$.

Next, we prove (37). Using Theorem 2, we have

$$\|G(s) - \hat{G}(s; P)\|_{\mathcal{H}_2} \leq \|\hat{\Xi}(s; P)\|_{\mathcal{H}_\infty} \|\bar{P}^\top \bar{P}X(s)\|_{\mathcal{H}_2}$$

where $\hat{\Xi}$ and X are defined as in (30). Note that the first condition in (35) implies that $\bar{P}B = 0$ for the feedthrough term of $\bar{P}X$. Furthermore, the second condition in (35) implies that

$$\text{tr}(\Phi) - \text{tr}(P\Phi P^\top) = \text{tr}(\bar{P}\Phi\bar{P}^\top) \leq \epsilon^2.$$

Thus

$$\begin{aligned} \|\bar{P}^\top \bar{P}X(s)\|_{\mathcal{H}_2} &= \sqrt{\text{tr}(\bar{P}^\top \bar{P}A\mathcal{W}A^\top \bar{P}^\top \bar{P})} \\ &= \sqrt{\text{tr}(\bar{P}\Phi\bar{P}^\top)} \leq \epsilon \end{aligned}$$

is ensured; see [38] for the calculation of the \mathcal{H}_2 -norm.

In what follows, we prove that

$$\|\hat{\Xi}(s; P)\|_{\mathcal{H}_\infty} < \gamma \quad (38)$$

by virtue of (33) and the first condition in (35). Note that, because $A + A^\top \prec \mathcal{O}_n$ owing to the I_n -dissipativity of Σ , there always exists some $\gamma > 0$ such that (33). Furthermore, the feedthrough term $C\Pi$ of $\hat{\Xi}$ is equal to zero because the first

condition in (35) holds. Thus, from the bounded real lemma [38], it follows that $\|\hat{\Xi}\|_{\mathcal{H}_\infty} < \gamma$ if there exists $\hat{V} = \hat{V}^\top \succ \mathcal{O}_{\hat{n}}$ such that

$$\hat{V}\hat{A} + \hat{A}^\top\hat{V} + \gamma^{-1} \left\{ \hat{V}\tilde{P}\tilde{P}^\top\hat{V} + \hat{C}^\top\hat{C} \right\} \prec \mathcal{O}_{\hat{n}}, \quad (39)$$

where $\tilde{P} := P + P\Pi$. We suppose that the explicit solution is $\hat{V} = I_{\hat{n}}$. Given that $\hat{C} = CP^\top = C\tilde{P}^\top$, (39) with $\hat{V} = I_{\hat{n}}$ becomes

$$\tilde{P} \{A + A^\top + \gamma^{-1}(I_n + C^\top C)\} \tilde{P}^\top \prec \mathcal{O}_{\hat{n}}.$$

Note that this strict inequality is ensured by (33) because \tilde{P} has full row rank as shown in Lemma 2. Hence, (38) follows for any P satisfying the first condition in (35). Finally, by $X(0) = 0$ in (29), $G(0) = \hat{G}(0; P)$ is proven. ■

Theorem 3 shows that the singular perturbation approximation admits the a priori error bound in (37). Note that the value of γ in (37) corresponds to an upper bound for the gain of the state-to-output mapping of the singular perturbation model.

Furthermore, to find $P \in \mathcal{P}^{\hat{n} \times n}$ with an appropriate dimension \hat{n} satisfying (35), we can use the following procedure for a prescribed ϵ : First, we find the set $\{(\lambda_i, v_i)\}_{i \in \{1, \dots, n\}}$ of all eigenpairs of Φ in (36), where it is assumed without loss of generality that $\lambda_i \geq \lambda_{i+1}$ and $\|v_i\| = 1$. Next, we find $k \in \{1, \dots, n\}$ such that

$$\sum_{i=k+1}^n \lambda_i \leq \epsilon^2, \quad (40)$$

and construct $V_k = [v_1, \dots, v_k] \in \mathcal{P}^{n \times k}$. Note that k is determined as being compatible with the prescribed ϵ . Finally, by the Gram-Schmidt process, we derive $P \in \mathcal{P}^{\hat{n} \times n}$ such that

$$\text{im}(P^\top) = \text{im}([V_k, B, C^\top]).$$

This projection matrix P produces a singular perturbation model having the dimension of $\hat{n} = \text{rank}([V_k, B, C^\top])$.

It is worth noting that the resultant approximation error is related to the sum of neglected eigenvalues of Φ as shown in (40). The major significance of Theorem 3 is the theoretical revelation that ϵ , which corresponds to the threshold of neglected eigenvalues of Φ , can be used as a design parameter to regulate the approximating quality as well as an appropriate dimension of resultant singular perturbation models.

Remark 4: By replacing the matrices B , C and D in (15) with empty matrices, we notice that the dissipation inequality in (14) can be reduced to the standard Lyapunov inequality

$$A^\top V + VA \prec \mathcal{O}_n. \quad (41)$$

Thus all the results derived above can be straightforwardly applied as a stability-preserving model reduction method for any stable system that admits a Lyapunov function as f_V in (16). Note that the first condition in (35) is not necessary to guarantee the stability of approximants, but it is necessary to prove the a priori error bound in (37).

Remark 5: The results derived in Sections III-B and III-C are based on the input-to-state mapping approximation. This can be verified from the facts that $P + P\Pi$ being a factor of \hat{A} in (21) is compatible with $\hat{B} = (P + P\Pi)B$, and that

$\bar{P}^\top \bar{P} X$ in (29) consists of \bar{P} , A and B . In fact, through dual arguments, it is possible to derive the same results based on the state-to-output mapping approximation by replacing (A, B, C) with (A^\top, C^\top, B^\top) .

IV. STRUCTURED CONTROLLER REDUCTION FOR DISTRIBUTED CONTROL SYSTEMS

A. Singular Perturbation Approximation of Distributed Control Systems

In this subsection, the controller reduction for distributed control systems described in Section II will be realized through the singular perturbation approximation associated with a structured projection matrix $P \in \mathcal{P}^{\hat{n} \times n}$. By taking $x = [x_0^\top, x_1^\top, \dots, x_L^\top]^\top$ as the state variable, it is verified that the distributed control system $(\Sigma_0, \{\Sigma_l\}_{l \in \mathbb{L}})$ in (1) can be described by Σ in (4) with the structured system matrices

$$\begin{aligned} A &= \begin{bmatrix} A_0 + b_{0,\mathbb{L}} \text{dg}(d_l)_{l \in \mathbb{L}} c_{0,\mathbb{L}} & b_{0,\mathbb{L}} \text{dg}(c_l)_{l \in \mathbb{L}} \\ \text{dg}(b_l)_{l \in \mathbb{L}} c_{0,\mathbb{L}} & \text{dg}(A_l)_{l \in \mathbb{L}} \end{bmatrix}, \\ B &= \begin{bmatrix} B_0 \\ 0 \end{bmatrix}, \quad C = [C_0 \quad 0], \quad D = D_0, \end{aligned} \quad (42)$$

where $b_{0,\mathbb{L}} := [b_{0,1}, \dots, b_{0,L}]$ and $c_{0,\mathbb{L}} := [c_{0,1}^\top, \dots, c_{0,L}^\top]^\top$. In what follows, we denote the dimensions of the plant Σ_0 and each controller Σ_l by n_0 and n_l , respectively, and we let $n := n_0 + \sum_{l=1}^L n_l$.

In general, the signal communication structure is destroyed by the direct application of singular perturbation approximations associated with an unstructured $P \in \mathcal{P}^{\hat{n} \times n}$ to the above structured system. In fact, the same difficulty is confronted by most traditional model reduction methods such as the balanced truncation, the Hankel-norm approximation, and the Krylov subspace methods [38].

Moreover, it is not clear whether there exists some P such that the singular perturbation approximation of Σ with (42) retains its structure. In view of this, we show in the following lemma that the singular perturbation approximation achieves the structured controller reduction by imposing a specific structure on P . It should be emphasized that this is nontrivial because the singular perturbation model $\hat{\Sigma}$ in (9) includes the matrix inverse $(\bar{P} A \bar{P}^\top)^{-1}$, which possibly becomes a dense matrix even if some sparse structure is imposed on $P \in \mathcal{P}^{\hat{n} \times n}$.

Lemma 3: Let a distributed control system $(\Sigma_0, \{\Sigma_l\}_{l \in \mathbb{L}})$ in (1) be given, and describe it by Σ in (4) with the system matrices in (42). Let $\hat{\Sigma}_l$ be the singular perturbation model of the controller Σ_l associated with $p_l \in \mathcal{P}^{\hat{n}_l \times n_l}$. Then, the singular perturbation model $\hat{\Sigma}$ in (9) associated with

$$P = \text{dg}(I_{n_0}, p_1, \dots, p_L) \in \mathcal{P}^{\hat{n} \times n}, \quad \hat{n} := n_0 + \sum_{l=1}^L \hat{n}_l \quad (43)$$

coincides with the approximate model $(\Sigma_0, \{\hat{\Sigma}_l\}_{l \in \mathbb{L}})$ in (2).

Proof: For each $l \in \mathbb{L}$, let $\bar{p}_l \in \mathcal{P}^{(n_l - \hat{n}_l) \times n_l}$ such that $[p_l^\top, \bar{p}_l^\top]^\top \in \mathbb{R}^{n_l \times n_l}$ is unitary. To prove the claim, it suffices

to show that $\hat{\Sigma}$ in (9) is given by the system matrices

$$\begin{aligned} \hat{A} &= \begin{bmatrix} A_0 + b_{0,\mathbb{L}} \text{dg}(\hat{d}_l)_{l \in \mathbb{L}} c_{0,\mathbb{L}} & b_{0,\mathbb{L}} \text{dg}(\hat{c}_l)_{l \in \mathbb{L}} \\ \text{dg}(\hat{b}_l)_{l \in \mathbb{L}} c_{0,\mathbb{L}} & \text{dg}(\hat{A}_l)_{l \in \mathbb{L}} \end{bmatrix}, \\ \hat{B} &= \begin{bmatrix} B_0 \\ 0 \end{bmatrix}, \quad \hat{C} = [C_0 \quad 0], \quad \hat{D} = D_0, \end{aligned} \quad (44)$$

where

$$\begin{aligned} \hat{A}_l &:= p_l A_l p_l^\top + p_l A_l \pi_l A_l p_l^\top, & \hat{b}_l &:= (p_l + p_l A_l \pi_l) b_l, \\ \hat{c}_l &:= c_l (p_l^\top + \pi_l A_l p_l^\top), & \hat{d}_l &:= d_l + c_l \pi_l b_l \end{aligned}$$

with $\pi_l := -\bar{p}_l^\top (\bar{p}_l A_l \bar{p}_l^\top)^{-1} \bar{p}_l$. Note that the structure of P in (43) allows for its orthogonal complement to have the form

$$\bar{P} = [0 \quad \text{dg}(\bar{p}_1, \dots, \bar{p}_L)] \in \mathcal{P}^{(n - \hat{n}) \times n}. \quad (45)$$

Thus, because $\bar{P} B = 0$ and $C \bar{P}^\top = 0$, it readily follows that

$$\hat{B} = P B, \quad \hat{C} = C P^\top, \quad \hat{D} = D.$$

This proves the claim for \hat{B} , \hat{C} , and \hat{D} in (44). In addition, because of the specific structures of P and \bar{P} , it follows that

$$\begin{aligned} P A P^\top &= \begin{bmatrix} A_0 + b_{0,\mathbb{L}} \text{dg}(d_l)_{l \in \mathbb{L}} c_{0,\mathbb{L}} & b_{0,\mathbb{L}} \text{dg}(c_l p_l^\top)_{l \in \mathbb{L}} \\ \text{dg}(p_l b_l)_{l \in \mathbb{L}} c_{0,\mathbb{L}} & \text{dg}(p_l A_l p_l^\top)_{l \in \mathbb{L}} \end{bmatrix}, \\ P A \bar{P}^\top &= \begin{bmatrix} b_{0,\mathbb{L}} \text{dg}(c_l \bar{p}_l^\top)_{l \in \mathbb{L}} \\ \text{dg}(p_l A_l \bar{p}_l^\top)_{l \in \mathbb{L}} \end{bmatrix}, \\ \bar{P} A P^\top &= [\text{dg}(\bar{p}_l b_l)_{l \in \mathbb{L}} c_{0,\mathbb{L}} \quad \text{dg}(\bar{p}_l A_l p_l^\top)_{l \in \mathbb{L}}], \\ \bar{P} A \bar{P}^\top &= \text{dg}(\bar{p}_l A_l \bar{p}_l^\top)_{l \in \mathbb{L}}. \end{aligned}$$

Note that

$$(\bar{P} A \bar{P}^\top)^{-1} = \text{dg}((\bar{p}_l A_l \bar{p}_l^\top)^{-1})_{l \in \mathbb{L}}. \quad (46)$$

Thus, $\hat{A} = P A P^\top - P A \bar{P}^\top (\bar{P} A \bar{P}^\top)^{-1} \bar{P} A P^\top$ is given by (44). ■

Lemma 3 shows that, if a block-diagonal structure as in (43) is imposed on P , then the singular perturbation approximation of Σ with (42) leads to the singular perturbation approximation of each of controllers Σ_l . This result is derived from the fact that the particular structure of P in (43) is compatible with A in (42), and leads to the block-diagonal structure of the matrix inverse $(\bar{P} A \bar{P}^\top)^{-1}$ as shown in (46). In this sense, the structured singular perturbation approximation has good compatibility with the controller reduction problem for distributed control systems.

It should be noted that a result similar to Lemma 3 is available even if some communication among controllers is allowed. More specifically, if

$$\text{im}([f_l, g_l^\top]) \subseteq \text{im}(p_l^\top), \quad \forall l \in \mathbb{L},$$

where f_l and g_l are the parameters of the controller Σ_l in (3), then the singular perturbation model $\hat{\Sigma}$ in (9) associated with P in (43) coincides with the approximation model $(\Sigma_0, \{\hat{\Sigma}_l\}_{l \in \mathbb{L}})$ in (2) that retains the communication structure among controllers.

B. Error Analysis for Structured Controller Reduction

In this subsection, we develop a structured controller reduction method for distributed control systems utilizing the dissipativity-preserving model reduction in Section III. To this end, we introduce the notion of dissipative system interconnection [10]–[12].

Definition 2: A distributed control system $(\Sigma_0, \{\Sigma_l\}_{l \in \mathbb{L}})$ in (1) is said to be *neutral associated with* $(V_0, \{V_l\}_{l \in \mathbb{L}})$ if the following conditions hold:

- The internal system

$$\Sigma_0^{(z,w)} : \begin{cases} \dot{x}_0 = A_0 x_0 + b_{0,\mathbb{L}} w \\ z = c_{0,\mathbb{L}} x_0 \end{cases} \quad (47)$$

is V_0 -dissipative with respect to Q_0 .

- Each controller Σ_l is V_l -dissipative with respect to Q_l .
- The set of supply functions defined as in (17) satisfies

$$s_{Q_0}(z, w) + \sum_{l=1}^L s_{Q_l}(w_l, z_l) = 0 \quad (48)$$

where $z = [z_1^T, \dots, z_L^T]^T$ and $w = [w_1^T, \dots, w_L^T]^T$.

In systems theory, the equality in (48) is called the *neutrality condition* of dissipative system interconnection. Note that an interconnected system composed of dissipative subsystems is not necessarily dissipative, and the neutrality of interconnection is known as a natural condition to guarantee its dissipativity. Examples of neutral interconnection include the negative feedback interconnection of passive systems and the positive feedback interconnection of bounded real systems. To see this, let us consider the case where two dissipative systems are interconnected. Namely, consider the neutrality condition

$$s_{Q_1}(y_1, u_1) + s_{Q_2}(y_2, u_2) = 0$$

under the interconnection of $u_1 = y_2 \in \mathbb{R}^{m_1}$ and $u_2 = y_1 \in \mathbb{R}^{m_2}$. This condition is satisfied for the negative feedback interconnection of any passive systems, i.e., dissipative systems with respect to

$$Q_1 = \begin{bmatrix} 0 & I_{m_1} \\ I_{m_1} & 0 \end{bmatrix}, \quad Q_2 = - \begin{bmatrix} 0 & I_{m_2} \\ I_{m_2} & 0 \end{bmatrix}.$$

It is also satisfied for the positive feedback interconnection of bounded real systems, i.e., dissipative systems with respect to

$$Q_1 = \begin{bmatrix} -\gamma^2 I_{m_1} & 0 \\ 0 & I_{m_2} \end{bmatrix}, \quad Q_2 = \begin{bmatrix} -I_{m_2} & 0 \\ 0 & \gamma^2 I_{m_1} \end{bmatrix}.$$

Generalization to the interconnection of more than two systems straightforwardly follows from the same argument.

Furthermore, it can be verified with Lemma 1 that, by a coordinate transformation of each controller, any neutral distributed control system $(\Sigma_0, \{\Sigma_l\}_{l \in \mathbb{L}})$ can be transformed to one that is neutral associated with $(V_0, \{I_{n_l}\}_{l \in \mathbb{L}})$; namely, every controller Σ_l can be I_{n_l} -dissipative. Therefore, without loss of generality, we can assume that any neutral distributed control system is neutral associated with $(V_0, \{I_{n_l}\}_{l \in \mathbb{L}})$.

On the basis of these facts, we state the main theorem of this section. Combining all the results derived above, we develop a structured controller reduction method for distributed control systems.

Theorem 4: Let a distributed control system $(\Sigma_0, \{\Sigma_l\}_{l \in \mathbb{L}})$ in (1) be given, and suppose that it is neutral associated with $(V_0, \{I_{n_l}\}_{l \in \mathbb{L}})$. Describe $(\Sigma_0, \{\Sigma_l\}_{l \in \mathbb{L}})$ by Σ in (4) with the system matrices in (42), and let $\gamma > 0$ such that

$$VA + A^T V + \gamma^{-1}(V^2 + C^T C) \prec \mathcal{O}_n \quad (49)$$

where

$$V := \text{dg}(V_0, I_{n_1}, \dots, I_{n_L}). \quad (50)$$

Furthermore, let $\hat{\Sigma}_l$ be the singular perturbation model of each controller Σ_l associated with $p_l \in \mathcal{P}^{\hat{n}_l \times n_l}$. If P in (43) satisfies

$$\text{im}(c_l^T) \subseteq \text{im}(p_l^T), \quad \sum_{l=1}^L \{ \text{tr}([\Phi]_l) - \text{tr}(p_l[\Phi]_l p_l^T) \} \leq \epsilon^2 \quad (51)$$

where $[\Phi]_l \in \mathbb{R}^{n_l \times n_l}$ denotes the principal submatrix of Φ in (36) compatible with Σ_l , then the singular perturbation model $\hat{\Sigma}$ in (9) satisfies (37) and coincides with an approximate model $(\Sigma_0, \{\hat{\Sigma}_l\}_{l \in \mathbb{L}})$ in (2) that is neutral associated with $(V_0, \{I_{\hat{n}_l}\}_{l \in \mathbb{L}})$.

Proof: We prove the claim with an argument similar to the proof of Theorem 3. As shown in Lemma 3, the singular perturbation approximation of Σ associated with P in (43) exactly coincides with that of the controller Σ_l associated with p_l for each $l \in \mathbb{L}$. In addition, by using Theorem 1 with the first condition in (51), we can verify that the approximation of each I_{n_l} -dissipative Σ_l yields an $I_{\hat{n}_l}$ -dissipative $\hat{\Sigma}_l$ with respect to the same Q_l . Thus, the approximate model $(\Sigma_0, \{\hat{\Sigma}_l\}_{l \in \mathbb{L}})$ is neutral associated with $(V_0, \{I_{\hat{n}_l}\}_{l \in \mathbb{L}})$ whenever the original $(\Sigma_0, \{\Sigma_l\}_{l \in \mathbb{L}})$ is neutral associated with $(V_0, \{I_{n_l}\}_{l \in \mathbb{L}})$.

Next, we prove that (38) is ensured by (49). From (18), it follows that

$$\dot{f}_{V_0}(x_0) < s_{Q_0}(z, w), \quad \dot{f}_{I_{n_l}}(x_l) < s_{Q_l}(w_l, z_l), \quad l \in \mathbb{L}.$$

Thus, by (48), we obtain

$$\dot{f}_{V_0}(x_0) + \sum_{l=1}^L \dot{f}_{I_{n_l}}(x_l) < s_{Q_0}(z, w) + \sum_{l=1}^L s_{Q_l}(w_l, z_l) = 0.$$

This implies that $f_V(x)$ is a Lyapunov function of Σ . That is, Σ admits the Lyapunov function $f_V(x)$ in (16) for V in (50), owing to the neutrality associated with $(V_0, \{I_{n_l}\}_{l \in \mathbb{L}})$.

Since $VA + A^T V \prec \mathcal{O}_n$, there exists some $\gamma > 0$ such that (49). Noting that $\tilde{P}V = \hat{V}\tilde{P}$ where

$$\tilde{P} := P + P\Pi, \quad \hat{V} := \text{dg}(V_0, I_{\hat{n}_1}, \dots, I_{\hat{n}_L}),$$

we verify that (39) with the explicit solution \hat{V} becomes

$$\tilde{P} \{ VA + A^T V + \gamma^{-1}(V^2 + C^T C) \} \tilde{P}^T \prec \mathcal{O}_{\hat{n}},$$

where the strict inequality is ensured by (49). Hence, (38) follows. Finally, owing to the block-diagonal structure of P in (43), the second condition in (35) reduces to that in (51). Hence, the claim follows. ■

Theorem 4 shows that the structured controller reduction appropriately preserves the neutrality of dissipative control systems, which can be interpreted as the passivity of the controllers. Furthermore, the performance degradation of closed-loop systems can be evaluated in terms of the \mathcal{H}_2 -norm.

For a given distributed control system $(\Sigma_0, \{\Sigma_l\}_{l \in \mathbb{L}})$, which is assumed to be neutral associated with $(V_0, \{I_{n_l}\}_{l \in \mathbb{L}})$, Fig. 3 lists an algorithm to find P in (43) such that (51) holds for a prescribed $\epsilon \geq 0$. In this algorithm, the prescribed ϵ is used as a threshold for the eigenvalues of $[\Phi]_l$ to be neglected. The case is similar to that of the balanced truncation, in which the Hankel singular values to be neglected are determined according to a prescribed threshold.

The major advantage of the algorithm is that ϵ can be used as a design parameter to regulate the approximating quality of the structured controller reduction. Moreover, it automatically finds an appropriate dimension of each approximate controller by considering the dynamical behavior of the set of distributed controllers as well as the controlled plant. Note that such a determination of appropriate controller dimension cannot be achieved by the simple application of an existing model reduction method to each disconnected controller. This advantage will be demonstrated in a numerical example in Section IV-C below.

Additionally, a discussion on the conservativeness of the a priori error bound seems in order. Indeed, even though Theorem 4 provides a theoretically reasonable strategy to find $P \in \mathcal{P}^{\hat{n} \times n}$, the error bound in (37), which can be calculated before the approximation, may become conservative, especially in a large-scale setting. This is because no information on P and \bar{P} is taken into account to derive the upper bound of γ in (49).

To compensate for this weakness, we propose an efficient way to calculate the approximation error by utilizing the *cascaded* form of the error system denoted in Theorem 2. For a fixed $P \in \mathcal{P}^{\hat{n} \times n}$ satisfying the first condition in (35), let $\mathcal{X} \succeq \mathcal{O}_{\hat{n}}$ and $\mathcal{Y} \in \mathbb{R}^{\hat{n} \times n}$ be solutions of the Lyapunov and Sylvester equations

$$\begin{cases} \hat{A}\mathcal{X} + \mathcal{X}\hat{A}^\top + Z\mathcal{Y}^\top + \mathcal{Y}Z^\top = 0 \\ \hat{A}\mathcal{Y} + \mathcal{Y}A^\top + Z\mathcal{W} = 0, \end{cases} \quad (52)$$

where $Z := (P + P\Lambda\Pi)\bar{P}^\top\bar{P}A$, and $\mathcal{W} \succeq \mathcal{O}_n$ is the solution of (34). Since we only need to find the lower-dimensional solutions \mathcal{X} and \mathcal{Y} individually, the equations in (52) can be solved more efficiently than the Lyapunov equation with respect to the $(n + \hat{n})$ -dimensional error system, whose controllability gramian in the cascaded realization is given by

$$\begin{bmatrix} \mathcal{X} & \mathcal{Y} \\ \mathcal{Y}^\top & \mathcal{W} \end{bmatrix} \in \mathbb{R}^{(n+\hat{n}) \times (n+\hat{n})}.$$

Note that \mathcal{W} should be obtained in advance to find P ; see the algorithm in Fig. 3. Then, the approximation error in terms of the \mathcal{H}_2 -norm is calculated with

$$\|G(s) - \hat{G}(s; P)\|_{\mathcal{H}_2} = \sqrt{\text{tr}(\hat{C}\mathcal{X}\hat{C}^\top)}. \quad (53)$$

By the algorithm in Fig 3 in conjunction with this a posteriori error calculation, a solution to the problem of the structured controller reduction in Section II is provided as follows:

- (a) Prescribe the admissible error $\delta \geq 0$.
- (b) For a fixed $\epsilon \geq 0$, execute the algorithm in Fig 3 to find $P \in \mathcal{P}^{\hat{n} \times n}$ in (43) satisfying (51), where the controllability gramian $\mathcal{W} \succeq \mathcal{O}_n$ in (34) is obtained.

- (c) Find $\mathcal{X} \in \mathbb{R}^{\hat{n} \times \hat{n}}$ by solving the Lyapunov and Sylvester equations in (52).
- (d) Calculate the resultant approximation error $\|G - \hat{G}\|_{\mathcal{H}_2}$ from (53).
- (e) If the approximation error is not less than δ , then return to (b) after setting a smaller ϵ .

The efficiency of this controller reduction procedure is demonstrated through a numerical example in Section IV-C.

For the implementability of this procedure, we give an additional note on the computational cost of finding the controllability gramian \mathcal{W} . Even though the computation of \mathcal{W} possibly becomes time consuming in a large-scale setting, some effective methods for solving large-scale Lyapunov equations are available from the literature. For example, [39] utilizes a Krylov subspace method, known as a computationally efficient method for model reduction. As a similar approach, [40] and [41] develop approximate solution algorithms by explicitly considering the sparsity and low-rankness of system matrices.

Finally, we clarify the contribution of this paper in comparison with existing model reduction methods and its generalization to structured ones. In model reduction theory, it often becomes an issue that a resultant approximate model is possibly unstable even if the original system is stable. To guarantee the stability of approximants, a rigid transformation, such as a balancing transformation, is generally required [37], [38]. Even though such a transformation is reliable in the approximation of disconnected systems, it is not necessarily flexible for a generalization to interconnected system approximation. Indeed, in structured model reduction [42], [43], to ensure the stability of approximate models and the existence of an a priori error bound, we need to impose a block-diagonal structure on the solutions of linear matrix inequalities, whose feasibility is not always guaranteed.

Nonetheless, we have shown in Section III that a transformation based on a storage function is sufficient for the stability preservation in the singular perturbation approximation. This relaxation allows us to enhance its applicability to interconnected systems. Note, however, that an interconnected system does not always admit a disjoint Lyapunov function with respect to each subsystem, even if all subsystems are stable. Our success in developing the network structure-preserving model reduction method is the focus on the class of systems composed of the neutral interconnection of dissipative subsystems, which admits the Lyapunov function f_V in (16) for the block-diagonal matrix V in (50). In addition, deriving the factorization of error systems as in Theorem 2, which is valid for any structured projection matrix P , we have shown the existence of an a priori error bound in Theorem 4.

Remark 6: In Definition 2, it is supposed that the internal system $\Sigma_0^{(z,w)}$ admits the strict notion of dissipativity introduced in Definition 1. It should be noted that, even if the strict dissipativity of $\Sigma_0^{(z,w)}$ is replaced with a *weak* notion of dissipativity, called semidissipativity, all results derived in this section are still valid. The definition of semidissipativity is as follows: A linear system Σ in (4) is said to be *V-semidissipative with respect to Q* if (A, B) is controllable

Input: $\epsilon \geq 0$, $(\Sigma_0, \{\Sigma_l\}_{l \in \mathbb{L}})$ in (1) described by Σ in (4) with (42)
Output: P in (43) such that (51)

- 1: Find \mathcal{W} such that (34)
- 2: $\Phi \leftarrow A\mathcal{W}A^\top$
- 3: Find Λ and V such that $\text{dg}([\Phi]_l)_{l \in \mathbb{L}} V = V^\top \Lambda$
- 4: $[\tilde{\Lambda}, \mathcal{I}] = \text{sort}(\text{diag}(\Lambda), \text{'descend'})$
- 5: $\lambda \leftarrow \tilde{\Lambda}(1)$, $\mathcal{J} \leftarrow \mathcal{I}(1)$
- 6: **while** $\text{tr}(\Lambda) - \lambda > \epsilon^2$ **do**
- 7: $\lambda \leftarrow \lambda + \tilde{\Lambda}(|\mathcal{J}|)$, $\mathcal{J} \leftarrow \{\mathcal{J}, \mathcal{I}(i)\}$
- 8: **end while**
- 9: $N \leftarrow n_1 + \dots + n_L$, $\hat{N} \leftarrow |\mathcal{J}|$
- 10: $I = \text{eye}(N)$, $e_{\mathcal{J}} \leftarrow I(:, \mathcal{J})$
- 11: Find $\tilde{P} \in \mathcal{P}^{\hat{N} \times \hat{N}}$ such that $\text{im}(\tilde{P}^\top) = \text{im}([Ve_{\mathcal{J}}, \text{dg}(c_l^\top)_{l \in \mathbb{L}}])$
- 12: $P \leftarrow \text{dg}(I_{n_0}, \tilde{P})$

Fig. 3. MATLAB style pseudo-code to find projection matrix for singular perturbation approximation. (The fourth line sorts the diagonal elements of Λ in descending order, and stores the ordered values and indices to $\tilde{\Lambda}$ and \mathcal{I} . The tenth line stores the column vectors of the N -dimensional identity matrix compatible with \mathcal{J} to $e_{\mathcal{J}}$, which is used to select the associated column vectors of V in the eleventh line.)

and there exists $V = V^\top \succeq \mathcal{O}_n$ such that (14) holds with nonstrict inequality. Similarly to the strict notion of dissipativity, the notion of a V -semidissipative system satisfies (18) with nonstrict inequality.

C. Numerical Example

In this subsection, we demonstrate the efficiency of our method through a numerical example. Let us consider the following mass-spring-damper system

$$\begin{cases} M\ddot{\zeta} + R\dot{\zeta} + K\zeta = Fw \\ z = H\dot{\zeta} \end{cases} \quad (54)$$

where $M \succ \mathcal{O}_\nu$ denotes a diagonal mass matrix, $R \succ \mathcal{O}_\nu$ denotes a diagonal damper matrix, $K \succ \mathcal{O}_\nu$ denotes a spring stiffness matrix, $F \in \mathbb{R}^{\nu \times m}$ denotes a matrix describing actuator allocation, and $H \in \mathbb{R}^{q \times \nu}$ denotes a matrix describing sensor allocation. This second-order system is often used as a primary model of flexible mechanical systems in vibration suppression control [13], [44] and in rotor dynamics for power system stabilization [45], [46].

Let $x_0 := [\zeta^\top, \dot{\zeta}^\top]^\top \in \mathbb{R}^{2\nu}$ be the state variable of Σ_0 in (1). Then, we have the 2ν -dimensional internal system $\Sigma_0^{(z,w)}$ in (47), with

$$A_0 = \begin{bmatrix} 0 & I_\nu \\ -M^{-1}K & -M^{-1}R \end{bmatrix}, \quad b_{0,\mathbb{L}} = \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix}, \\ c_{0,\mathbb{L}} = \begin{bmatrix} 0 & H \end{bmatrix}.$$

It is shown in [13] that this $\Sigma_0^{(z,w)}$ is passive as long as the input and output are collocated, namely $F = H^\top$, and $(A_0, b_{0,\mathbb{L}})$ is controllable. More specifically, for $V_0 = \text{dg}(K^{-1}, M^{-1})$, this $\Sigma_0^{(z,w)}$ is V_0 -semidissipative with respect to Q in (19); see Remark 6 about semidissipativity. In what follows, by supposing that the system in (54) has some spatial distribution, we consider the distributed control of vibration suppression for (54). Such distributed control is reasonable in the sense that sensor and actuator allocation is often limited by complying

with some physical restrictions, as in vibration suppression for bridges [47].

Let us consider a case where 125 mass components are coupled. For this 250-dimensional passive system, we specify the coefficient matrices in (54) as

$$M = \frac{1}{5}I_{125}, \quad F = H^\top = \text{dg}(e_1, \dots, e_1), \\ K = \frac{1}{2} \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 2 \end{bmatrix}, \quad (55) \\ R = \frac{1}{4} \text{dg}(I_{25}, rI_{25}, r^2I_{25}, r^3I_{25}, r^4I_{25}),$$

where $r = 3/10$ and e_1 denotes the first column of I_{25} . Furthermore, we take the evaluated output as $y = z \in \mathbb{R}^5$. This system is depicted in Fig. 4, where $w = [w_1, \dots, w_5]^\top$, $y = [y_1, \dots, y_5]^\top$, $z = [z_1, \dots, z_5]^\top$, and $\zeta = [\zeta_1, \dots, \zeta_{125}]^\top$.

For this plant Σ_0 , we construct a set of passive controllers Σ_l for $l \in \{1, \dots, 5\}$. To this end, we apply the passive controller synthesis proposed in [14] to each of the truncated (disconnected) subplants Σ_0^l for $l \in \{1, \dots, 5\}$ shown in Fig. 4. As a result, we obtain a set of 50-dimensional original passive controllers. Indeed, such a disconnective approach to distributed control design is often taken in generator control for power networks. The outputs with and without control are shown in Fig. 5 by the solid and dot-dashed lines, respectively, where the initial condition $x_0(0)$ of the plant is random. We can see from this figure that the convergence rate of the system outputs with and without controllers becomes higher in the order of y_5 to y_1 . This comes from the gradual variation of the damping coefficients of R in (55).

Next, we reduce the dimension of each controller by using our structured controller reduction method while keeping the behavior of the closed-loop system. To guarantee the performance for *any* initial condition $x_0(0)$, we apply the dual coun-

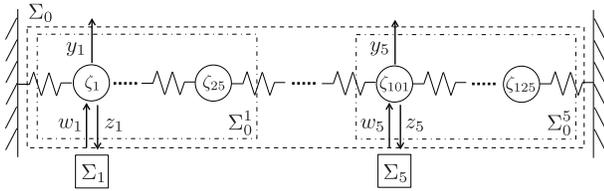


Fig. 4. Mass-spring-damper system with distributed passive control.

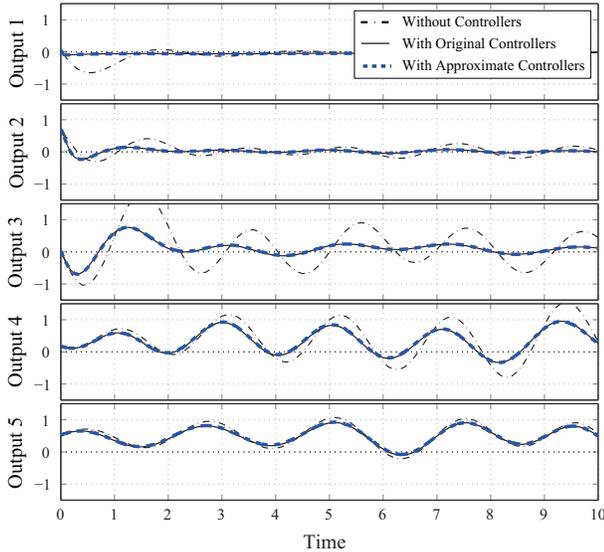


Fig. 5. Output signals of passive mass-spring-damper system.

terpart of Theorem 4 to this control system. More specifically, to approximate the state-to-output mapping defined by (A, C) in (42) with $C_0 = c_{0,L}$, we use Theorem 4 by replacing (A, B) with (A^T, C^T) . Furthermore, we regard B_0 in (42) as I_n , which reflects arbitrary initial condition.

In Fig. 6, plots of the eigenvalues of $[\Phi]_l$ in (51) are shown for each $l \in \{1, \dots, 5\}$. From this figure, we can see that the decay rate of the eigenvalues becomes faster in the order of $[\Phi]_5$ to $[\Phi]_1$. This characteristic indicates that the dimension of passive controllers with lower indices, e.g., Σ_1 and Σ_2 , can be more significantly reduced with a small approximation error.

We implement the controller reduction procedure proposed in Section IV-B. Let us prescribe the admissible error by $\delta = 0.47$, which corresponds to the 5% relative \mathcal{H}_2 -error. In Fig. 7, against each value of ϵ , we plot the resultant dimensions of the approximate controllers $\hat{\Sigma}_l$ for $l \in \{1, \dots, 5\}$. Furthermore, by the line with squares, in Fig. 8, we plot the resultant relative errors, i.e., $\|G - \hat{G}\|_{\mathcal{H}_2} / \|G\|_{\mathcal{H}_2}$, which is the value from (53). These figures show that the dimension of the approximate controllers increases, and the approximation error appropriately decreases, as ϵ decreases. Note that the dimension of each approximate controller is automatically determined for each value of ϵ with the dynamics of the controllers as well as that of the controlled plant being explicitly considered. This result thus confirms that the value of ϵ , which corresponds to the threshold of the eigenvalues of $[\Phi]_l$, successfully captures the degree of performance degradation.

When $\epsilon = 5.3$, the original 50-dimensional passive con-

trollers Σ_l for $l \in \{1, \dots, 5\}$ are reduced to 5-, 3-, 9-, 13- and 24-dimensional versions $\hat{\Sigma}_l$, and the resultant approximation error is $\|G - \hat{G}\|_{\mathcal{H}_2} = 0.34$, which is less than the prescribed δ . The output of the closed-loop system with the approximate passive controllers is overplotted in Fig. 5 with the dashed lines. We can see that the dimension of the passive controllers is appropriately reduced almost without affecting the behavior of the closed-loop system.

For comparison, we show results when using some other model/controller reduction methods. As standard model reduction methods, not directly dealing with the structured controller reduction problem, we implement a passivity-preserving Krylov subspace method [48], which is known as a major reduction method for passive RLC circuits, and the balanced residualization [35], [37], which is implemented as a singular perturbation approximation of the balanced realization, to each disconnected controller. Furthermore, we implement a structured balanced truncation method [42] that maintains the interconnection structure of systems by considering the plant and controller dynamics.

In Fig. 8, we overplot the resultant relative approximation error of the closed-loop system from each method, along with plots of the relative error between the original and approximate controllers in Fig. 9, where the \mathcal{H}_∞ -error is calculated because the balanced residualization possibly yields an approximate model with a nonzero feedthrough term which brings on the unbounded \mathcal{H}_2 -norm. To make the comparison fair, we give the dimensions of approximate controllers such that the total number of controller dimensions coincides with that resulting from our method. More specifically, we give the controller dimension for the Krylov subspace method so that each controller has an identical dimension up to the difference of a residue modulo 5. For the balanced residualization, each controller dimension is determined by the number of states compatible with the least sum of the Hankel singular values for all controllers. In a similar manner, we determine the controller dimension for the structured balanced truncation method using the structured Hankel singular values [42].

Inspecting the lines with diamonds in Figs. 8 and 9, we see that the decreasing rate of closed-loop system approximation errors from the Krylov subspace method is lower, even though its performance of lower-dimensional approximation is better than that of the other methods. This trend results from the fact that an appropriate dimension of controllers is not systematically determined by the Krylov subspace method in general. On the other hand, the balanced residualization, denoted by the lines with circles, gives the best approximation for disjoint controllers while being conducive to a precipitous variation of approximation errors arising in the closed-loop system. Note that the closed-loop stability is not guaranteed theoretically. This result demonstrates that the behavior of the closed-loop system is possibly affected by even a small error in the controller approximation. Finally, inspecting the lines with triangles in Figs. 8 and 9, we see that the decreasing rate of closed-loop system approximation errors from the structured balanced truncation is lower than ours while it gives better controller and closed-loop system approximation for higher-dimensional approximate controllers. Note that the

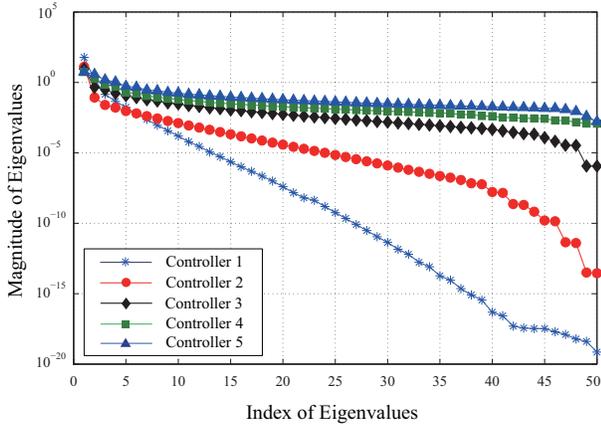


Fig. 6. Eigenvalues of principal submatrices $[\Phi]_l$ for $l \in \{1, \dots, 5\}$.

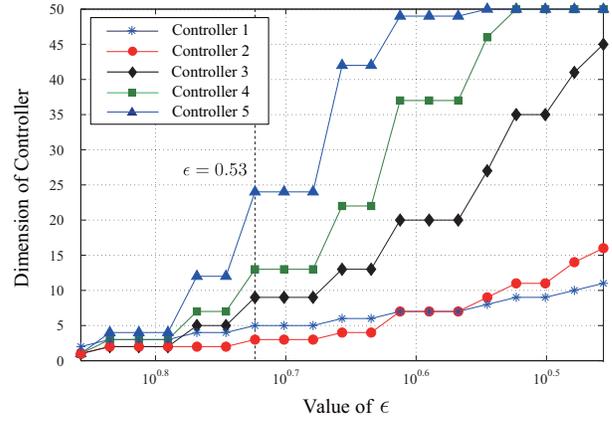


Fig. 7. Resultant dimension of approximate controllers.

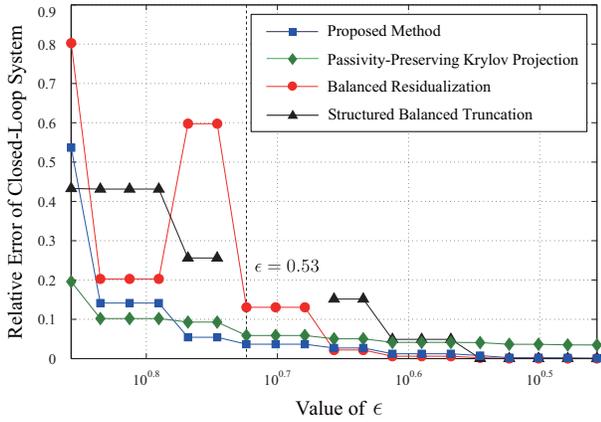


Fig. 8. Approximation error of closed-loop system versus value of ϵ .

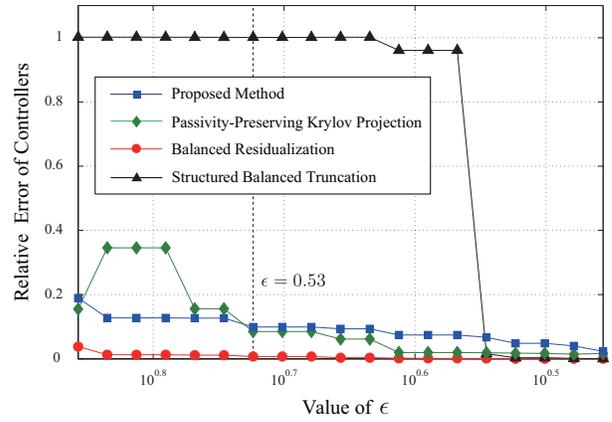


Fig. 9. Approximation error of controllers versus value of ϵ .

breakup of the line in Fig. 8 indicates that the resultant closed-loop systems are unstable. Indeed, unless there exist block-diagonally structured gramians solving Lyapunov equations or inequalities, closed-loop stability is not guaranteed in the structured balanced truncation. From this numerical example, we can affirm that our structured passive controller reduction method is more reliable in preserving closed-loop stability, i.e., passivity, with the dynamical behavior of the passive controllers and controlled plant being explicitly considered.

V. CONCLUSION

In this paper, we propose a structured controller reduction method for distributed control systems. As a fundamental tool to develop structured controller reduction, we first established dissipativity-preserving model reduction for general linear systems on the basis of a singular perturbation approximation. It was found that the singular perturbation approximation can be represented by a projection-like formula that enables us to characterize dissipativity preservation in a tractable manner and to derive a novel factorization of error systems. This error system factorization further provides a remarkable insight that the resultant approximation error is related to the sum of neglected eigenvalues of an index matrix.

Then, utilizing the dissipativity-preserving model reduction, we developed a structured controller reduction method by focusing on dissipative system interconnection. The major significance is that it not only preserves the spatial distribution of dissipative controllers but it also provides an a priori \mathcal{H}_2 -error bound for the structured controller reduction. The efficiency of our method was demonstrated through a numerical example of vibration suppression control for spatially distributed plants. Via a comparison with some existing model/controller reduction methods, it was shown that our method can produce more reliable approximate distributed controllers, whose dimension is automatically determined by an eigenvalue analysis, while considering the dynamical behavior of the set of original controllers as well as a controlled plant.

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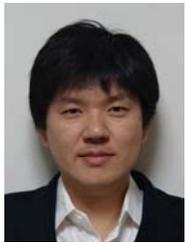


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