Clustered Model Reduction of Interconnected Second-Order Systems and Its Applications to Power Systems

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Outline

Introduction: Why clustered model reduction?

Clustered Model Reduction Theory
  - Interconnected first-order systems
  - Extension to second-order networks

Application to power networks

Conclusion
Control of Large-Scale Networks

Power Networks
- Tokyo area: **20 million** houses
  - Instability may be caused by renewables such as PV, wind

Model reduction is one prospective approach
- Center of Tokyo area: **5 million** cars
  - Heavy traffic jam

How to manage?
Standard Model Reduction Framework

\[ \Sigma : \begin{align*}
    \dot{x} &= Ax + Bu \\
y &= Cx \\
x &\in \mathbb{R}^n
\end{align*} \]

\[ P x = \hat{x} \]

\[ \hat{\Sigma} : \begin{align*}
    \dot{\hat{x}} &= P A P^\dagger \hat{x} + P B u \\
\hat{y} &= C P^\dagger \hat{x} \\
\hat{x} &\in \mathbb{R}^{\hat{n}} , \quad \hat{n} < n
\end{align*} \]

Main goal: Find \( P \) such that \( \| y - \hat{y} \| \) is small enough

+ stability of error systems, error analysis, low computation cost

Standard methods:

- Balanced truncation, Hankel norm approximation
  - error bound, stability preservation \( \smile \) high computational cost \( \frown \)
- Krylov projection
  - lower computation cost \( \smile \) possibly unstable model, no error bound \( \frown \)
Application of Standard Methods to Network Systems

**Drawback:** Network structure is lost through reduction

**Network system**

\[ \Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \]

**Reduced model**

\[ \hat{\Sigma} : \begin{cases} \dot{\hat{x}} = \hat{P}A\hat{P}^\dagger \hat{x} + \hat{P}Bu \\ \hat{y} = \hat{C}\hat{P}^\dagger \hat{x} \end{cases} \]

\[ P \in \mathbb{R}^{\hat{n} \times n}, \quad \hat{n} < n \]

Dense matrix

Sparse 😊

Dense 😞
Clustered Model Reduction

Network system

\[ \Sigma : \begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*} \]

Cluster state

\[ x[l] \]

Aggregated model

\[ \hat{\Sigma} : \begin{align*}
\dot{\hat{x}} &= PAP^\dagger \hat{x} + PBu \\
\hat{y} &= CP^\dagger \hat{x}
\end{align*} \]

Aggregated state

\[ \hat{x}[l] = p[l]x[l] \]

Sparse ☺

Preservation of network structure among clusters ☺
Why Clustered Model Reduction?

Gene Network  [Mochizuki et al., J. Theoretical Biology (2010)]

Clustered model reduction

Extract essential structure to study mechanism of functions
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System Description (First-Order Subsystems)

[Definition] Bidirectional Network

\[ \dot{x} = Ax + Bu \quad \text{with} \quad A = \{a_{i,j}\} \in \mathbb{R}^{n \times n} \quad \text{and} \quad B = \{b_i\} \in \mathbb{R}^n \]

is said to be *bidirectional network* if \( A \) is symmetric and stable.

Reaction-diffusion systems:  

\[ \dot{x}_i = -r_i x_i + \sum_{j=1, j \neq i}^{n} a_{i,j} (x_j - x_i) + b_i u \]
Clustered Model Reduction Problem

[Problem] Given $\epsilon \geq 0$, find a cluster set $\{I_{[l]}\}_{l \in \mathbb{L}}$ such that

$$\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty} \leq \epsilon$$

where $g(s) := (sI_n - A)^{-1}B$ and $\hat{g}(s) := P^T(sI_{\hat{n}} - PAP^T)^{-1}PB$.

Bidirectional network

$$\Sigma : \dot{x} = Ax + Bu$$

Cluster state $x_{[l]}$

Sparse 😊

Aggregated model

$$\hat{\Sigma} : \begin{cases} \dot{\xi} = PAP^T\xi + PBu \\ \hat{x} = P^T\xi \end{cases}$$

Aggregated state

$$\xi_{[l]} = p_{[l]}x_{[l]}$$

Sparse 😊
How to Formulate Reducibility?

Bidirectional network \( \dot{x} = Ax + Bu \)

50 nodes, nonzero \( a_{i,j} \) is randomly chosen from \((0, 1)\)

[Definition] Reducible cluster

A cluster \( \mathcal{I}_u \) is said to be **reducible** if under any input signal \( u(t) \)

\[ x_i(t) \equiv x_j(t), \quad \forall i, j \in \mathcal{I}_u. \]
**Positive Tridiagonalization**

**Lemma** For every bidirectional network \((A, B)\), there exists a unitary \(H \in \mathbb{R}^{n \times n}\) such that \((\tilde{A}, \tilde{B}) = (H^T A H, H^T B)\) has the following structure.

Bidirectional network \((A, B)\)

\[
\dot{x} = Ax + Bu, \quad A = A^T
\]

(not necessarily positive)

Positive tridiagonal realization \((\tilde{A}, \tilde{B})\)

\[
\begin{align*}
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\vdots \\
\tilde{x}_n
\end{bmatrix} &= \begin{bmatrix}
\alpha_1 & \beta_1 \\
\beta_1 & \alpha_2 & \beta_2 \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & \beta_{n-1} & \alpha_n
\end{bmatrix} \begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\vdots \\
\tilde{x}_n
\end{bmatrix}
\end{align*}
\]

\[
\tilde{B} = \begin{bmatrix}
\beta_0 \\
0 \\
& \ddots \\
& & 0
\end{bmatrix}
\]

\(\checkmark \beta_i \geq 0\)
Reducibility Characterization

Bidirectional network \((A, B)\)

\[
A = \begin{bmatrix}
-(6 + 1) & 2 & 2 & 1 & 1 \\
2 & -2 & 0 & 0 & 0 \\
2 & 0 & -2 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & -1
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[\text{— graph Laplacian + diagonal} \]
Reducibility Characterization

Bidirectional network \((A, B)\)

\[ (\tilde{A}, \tilde{B}) : \text{positive tridiagonal realization} \]

\[ H : \text{transformation matrix} \]

Index matrix

\[ \Phi := H \text{diag}(−\tilde{A}^{-1} \tilde{B}) \]

Characterization in frequency domain

Equivalent characterization of cluster reducibility
\( \theta \)-Reducible Cluster Aggregation

**[Definition]** \( \theta \)-Reducibility of Clusters

A cluster \( \mathcal{I}_I \) is said to be \( \theta \)-reducible if

\[
\left\| \text{row}_i[\Phi] - \text{row}_j[\Phi] \right\|_\infty \leq \theta, \quad \forall i, j \in \mathcal{I}_I
\]

\( x_i(t) - x_j(t) \approx 0, \quad \forall u(t) \)

Similar behavior

**[Theorem]** If all clusters are \( \theta \)-reducible, then

\[
\| g(s) - \hat{g}(s) \|_{H_\infty} \leq \gamma \sqrt{\sum_{l=1}^{\hat{n}} |\mathcal{I}_I| (|\mathcal{I}_I| - 1)} \theta
\]

where \( \gamma := \| (PAP^T)^{-1} PA \| \).

\( \theta \): coarseness parameter

1000 nodes  \quad 47 clusters  \quad Extract essential cluster structure!

\( \theta = 1.5 \)

aggregation

about 5% error
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Formulation by Second-Order Systems

Second-order networks

\[ \Sigma : \ddot{x} + D \dot{x} + Kx = fu \]

\[ u \rightarrow x_1 \rightarrow x_2 \rightarrow x_4 \rightarrow x_3 \rightarrow x_5 \rightarrow \mathcal{I}_l \]

Aggregated model

\[ \hat{\Sigma} : \begin{cases} 
\ddot{\xi} + PDP^T \dot{\xi} + PKP^T \xi = Pf \xi \\
\hat{x} = P^T \xi 
\end{cases} \]

\[ PX = \xi \]

[Problem] Given \( \epsilon \geq 0 \), find a cluster set \( \{ \mathcal{I}_l \}_{l \in L} \) such that

\[ \| g(s) - \hat{g}(s) \|_{\mathcal{H}_\infty} \leq \epsilon \]

where \( g(s) := (s^2I_n + sD + K)^{-1}f \) and \( \hat{g}(s) := P^T(s^2\hat{I}_n + sPDP^T + PKP^T)^{-1}Pf \).
Extension to Second-Order Networks

First-order representation (2n-dim. system) \( X := \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \)

\[ \sum : \begin{cases} \dot{X} = AX + Bu \\ x = CX \end{cases} \]

where \( A := \begin{bmatrix} 0 & I_n \\ -K & -D \end{bmatrix} \)

\( B := \begin{bmatrix} 0 \\ f \end{bmatrix} \)

\( C := \begin{bmatrix} I_n & 0 \end{bmatrix} \)

Index matrix \( \Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \) w.r.t. \{ position \( x \), velocity \( \dot{x} \) \}

[Definition] A cluster \( \mathcal{I}_l \) is said to be \( \theta \)-reducible if

\[ ||\text{row}_i[\Phi_k] - \text{row}_j[\Phi_k]||_{l_\infty} \leq \theta, \quad \forall i, j \in \mathcal{I}_l, \quad k \in \{1, 2\} \]

[Theorem] If all clusters are \( \theta \)-reducible, then

\[ ||g(s) - \hat{g}(s)||_{\mathcal{H}_\infty} \leq \gamma \sqrt{\sum_{l=1}^{n} \mathcal{I}_l |(\mathcal{I}_l) - 1| \theta} \]

where \( \gamma := \sqrt{2} \left\| P(s^2 I_{\hat{n}} + sPDPT + PKP^T)^{-1} [PK \quad PD] - [I_n \quad 0] \right\|_{\mathcal{H}_\infty} \).
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Numerical Example

Power network modeled by swing equation

Original network (300 nodes)  Agg. model (42 clusters)

Zoom up

Position trajectory of a mass

Relative error
\[ \frac{\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty}}{\|g(s)\|_{\mathcal{H}_\infty}} = 4.81\% \]
when \( \theta = 1.76 \)
Concluding Remarks

- **Clustered model reduction**
  - extract essential information on input-to-state mapping
  - application to power networks model by swing equation

- **Future works**
  - application to more realistic power networks
  - extension to nonlinear systems
  - application to control system design


Thank you for your attention!