

Paper

Clustered Model Reduction of Interconnected Second-Order Systems

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Abstract: In this paper, we propose a clustered model reduction method for interconnected second-order systems evolving over undirected networks, which we call second-order networks. In this model reduction method, network clustering, i.e., clustering of subsystems, is performed according to cluster reducibility, which is defined as a notion of weak controllability of local subsystem states. This paper clarifies that the cluster reducibility can be algebraically characterized for second-order networks through the controller-Hessenberg transformation of their first-order representation. By aggregating the reducible clusters, we obtain an approximate model that preserves an interconnection topology among clustered subsystems. Furthermore, we derive an \mathcal{H}_∞ -error bound of the state discrepancy caused by the cluster aggregation. Finally, the efficiency of the proposed method is demonstrated through an example of large-scale complex networks.

Key Words: Clustered model reduction; Interconnected second-order systems; Complex networks; Network clustering.

1. Introduction

Dynamical systems arising in science and engineering are generally modeled as interconnected systems. Examples of such interconnected systems include power networks, transportation networks, communication networks and so forth; see [1, 2] for an overview. Since their interconnection topology is often complex and large-scale, it is crucial to develop an approximation method for reducing their complexity [3]. In addition, it is more desirable to preserve specific system properties, such as stability, throughout the approximation. Especially for interconnected systems, the preservation of their interconnection topology is one of the most important issues to be addressed. In fact, this kind of network structure-preserving model reduction has the potential to significantly simplify the analysis as well as control of large-scale interconnected systems while capturing their essential properties of interest.

Several network structure-preserving model reduction methods can be found in the literature. For example, [4] has proposed a Krylov subspace method for interconnected systems, in which the Krylov projection of each subsystem is performed to preserve the interconnection topology among subsystems. However, this method requires a priori information on the partition of the whole system into subsystems. Furthermore, no theoretical error evaluation is provided there. As a similar approach, [5] has proposed a structured balanced truncation method for interconnected systems, in which the balanced truncation approximation is applied to each subsystem. However, the relation between the subsystem partition and the resultant approximation error is not theoretically discussed there.

Against this background, the authors have developed a clustered model reduction method for interconnected linear systems, in which first-order subsystems are coupled over large-scale networks [6, 7]. In this method, we introduce the notion of cluster reducibility, which is defined as the uncontrollability of disjoint subsets of state variables, called clusters, while providing an algorithm to find a set of reducible clusters. By aggregating the reducible clusters with suitable aggregation coefficients, we can construct an approximate model that preserves the interconnection topology among clusters as well as the stability of systems. Furthermore, we have performed an error evaluation in terms of the $\mathcal{H}_2/\mathcal{H}_\infty$ -norm.

In this paper, we consider generalizing our clustered model reduction method to interconnected second-order systems evolving over undirected networks, which we call second-order networks. In fact, many physical systems can be modeled by such interconnected second-order systems [8–12]. However, developing a model reduction method for second-order networks is not necessarily straightforward. This is because the application of usual model reduction as well as clustered model reduction may destroy their second-order structure, i.e., the resultant aggregated models cannot be interpreted as second-order networks. To resolve this matter, we are required to confine the class of aggregated models to those with the second-order structure of interest. In this paper, we clarify that such aggregated models can be obtained by applying a block-diagonally structured orthogonal projection to a first-order representation of second-order networks. Furthermore, it turns out that the cluster reducibility is algebraically characterized for second-order networks through the controller-Hessenberg transformation of their first-order representation. This transformation also leads to a novel frequency domain characterization of system controllability having good compatibility with an \mathcal{H}_∞ -error evaluation.

Finally, it should be noted that, even though this paper focuses on the clustered model reduction for linear systems in terms of the \mathcal{H}_∞ -norm, the development of such an approximation method is indeed fundamental to deal with nonlinear systems. This is because it is generally hard to approximate nonlinear systems directly, while guaranteeing stability preservation and performing an error analysis. One reasonable approach to a nonlinear system approximation is decoupling them into linear and nonlinear components, and then approximating the linear component while retaining the nonlinear one. In fact, this approach has been taken in [13, 14] to perform an error analysis for a nonlinear system approximation on the premise that an \mathcal{H}_∞ -model reduction method for linear systems is available. Thus, towards the systematic approximation of nonlinear second-order networks, it is crucial to develop a clustered model reduction method to construct an aggregated model whose approximation quality is specified in terms of the \mathcal{H}_∞ -norm.

The remainder of this paper is structured as follows: In Section 2, we provide theoretical results on the clustered model reduction for second-order networks. More specifically, in Section 2.1, we first formulate a clustered model reduction problem for second-order networks, and then we propose its solution method in Section 2.2, while providing a simple example that explains the intuition of clustered model reduction. In Section 3, we demonstrate the efficiency of the proposed solution method through an example of large-scale complex networks. Finally, concluding remarks are provided in Section 4.

Notation. We denote the set of real numbers by \mathbb{R} , the n -dimensional identity matrix by I_n , the i th column of I_n by e_i^n . For a set of natural numbers $\mathcal{I} \subseteq \{1, \dots, n\}$, let $e_{\mathcal{I}}^n \in \mathbb{R}^{n \times |\mathcal{I}|}$ denote the matrix composed of the column vectors of I_n compatible with \mathcal{I} . and the l_∞ -induced norm of a matrix

$M \in \mathbb{R}^{n \times m}$ is defined by

$$\|M\|_{l_\infty} := \max_{i \in \{1, \dots, n\}} \sum_{j=1}^m |M_{i,j}|$$

where $M_{i,j}$ denotes the (i, j) -element of M . Furthermore, we denote the block diagonal matrix having matrices M_1, \dots, M_n on its block diagonal by $\text{diag}(M_1, \dots, M_n)$. Finally, the \mathcal{H}_∞ -norm of a stable transfer matrix G is defined by

$$\|G(s)\|_{\mathcal{H}_\infty} := \sup_{\omega \in \mathbb{R}} \|G(j\omega)\|.$$

2. Theoretical Results

2.1 Clustered Model Reduction Problem for Second-Order Networks

In this paper, we study the following class of interconnected second-order systems:

$$\Sigma : \ddot{x} + D\dot{x} + Kx = fu \quad (1)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ denote the state variable and control input, respectively, $D = D^\top \in \mathbb{R}^{n \times n}$ and $K = K^\top \in \mathbb{R}^{n \times n}$ are assumed to be positive definite, and $f \in \mathbb{R}^n$. In the following, we refer to Σ in (1) as a *second-order network*. This class of systems represents interconnected mass-spring-damper systems evolving over undirected networks, which are typified by spatially discretized flexible beams [15] and linearized swing equations for power network models [12]. For simplicity of explanation, we only consider single-input systems while a generalization to multi-input systems can be done by taking an approach similar to [16]. Note that any second-order network Σ is stable owing to the positive definiteness of D and K ; see [11] for its stability characterization. We consider the system behavior around some equilibrium state of a nonlinear second-order network, and so we focus on the linearized system Σ .

Let us formulate a problem of clustered model reduction for Σ in (1). To this end, we first introduce a notion of network clustering [6, 7] for dynamical systems as follows:

Definition 1. Let $\mathbb{L} := \{1, \dots, L\}$. The family of an index set $\{\mathcal{I}_{[l]}\}_{l \in \mathbb{L}}$ is called a *cluster set*, each of whose elements is referred to as a cluster, if each element $\mathcal{I}_{[l]}$ is a disjoint subset of $\{1, \dots, n\}$ and satisfies $\bigcup_{l \in \mathbb{L}} \mathcal{I}_{[l]} = \{1, \dots, n\}$. Furthermore, an *aggregation matrix* compatible with $\{\mathcal{I}_{[l]}\}_{l \in \mathbb{L}}$ is defined by

$$P := \text{diag}(p_{[1]}, \dots, p_{[L]})\Pi \in \mathbb{R}^{L \times n} \quad (2)$$

where $p_{[l]} \in \mathbb{R}^{1 \times |\mathcal{I}_{[l]}|}$ such that $\|p_{[l]}\| = 1$, and the permutation matrix Π is defined as

$$\Pi := [e_{\mathcal{I}_{[1]}}, \dots, e_{\mathcal{I}_{[L]}}]^\top \in \mathbb{R}^{n \times n}, \quad e_{\mathcal{I}_{[l]}}^n \in \mathbb{R}^{n \times |\mathcal{I}_{[l]}|}.$$

Using the aggregation matrix P in (2), we define the aggregated model of Σ in (1) by

$$\hat{\Sigma} : \begin{cases} \ddot{\xi} + PDP^\top \dot{\xi} + PKP^\top \xi = Pfu \\ \hat{x} = P^\top \xi. \end{cases} \quad (3)$$

Each state of the aggregated model $\hat{\Sigma}$ is an approximant of the clustered states given by $(e_{\mathcal{I}_{[l]}}^n)^\top x \in \mathbb{R}^{|\mathcal{I}_{[l]}|}$. The trajectory of each state of $\hat{\Sigma}$ aims to trace the trajectory of a kind of centroid compatible with the clustered states of Σ . Note that the aggregated model $\hat{\Sigma}$ is stable for any P because PDP^\top and PKP^\top are also positive definite. In this paper, we address the following problem of clustered model reduction for second-order networks:

Problem 1. Consider a second-order network Σ in (1). Given a constant $\epsilon \geq 0$, find an aggregation matrix P in (2) such that the aggregated model $\hat{\Sigma}$ in (3) satisfies

$$\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty} \leq \epsilon \quad (4)$$

where

$$g(s) := (s^2 I_n + sD + K)^{-1} f, \quad \hat{g}(s) := P^\top (s^2 I_L + sPDP^\top + PKP^\top)^{-1} P f \quad (5)$$

denote the transfer functions of Σ and $\hat{\Sigma}$, respectively.

In Problem 1, we have formulated the problem of finding an aggregated model that satisfies an error bound in terms of the \mathcal{H}_∞ -norm. In traditional model reduction, such as the balanced truncation, the Krylov projection, and the Hankel norm approximation, each state of the resultant approximants is usually obtained as a linear combination of all states of the original system, i.e., the transformation matrix is a full matrix [3]. This clearly contrasts with our problem formulation, where P in (2) is block-diagonally structured.

2.2 Solution Method

2.2.1 Controllability Characterization via Controller-Hessenberg Transformation

In this subsection, we address the clustered model reduction problem from a viewpoint of local uncontrollability of subsystem states. To this end, we represent Σ in (1) by the first-order form

$$\Sigma : \begin{cases} \dot{X} = AX + Bu \\ x = CX \end{cases} \quad (6)$$

where $X := [x^\top, \dot{x}^\top]^\top \in \mathbb{R}^{2n}$, and

$$A := \begin{bmatrix} 0 & I_n \\ -K & -D \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \quad B := \begin{bmatrix} 0 \\ f \end{bmatrix} \in \mathbb{R}^{2n}, \quad C := [I_n \quad 0] \in \mathbb{R}^{n \times 2n}.$$

In the following, $N := 2n$ is used for convenience of notation. In control theory, Σ in (6) is said to be *controllable* if there exists an input function u such that the state X is moved from any initial state to any other final state in a finite time interval. One best-known characterization of controllability is the Kalman rank condition, i.e., Σ is controllable if and only if $[B, AB, \dots, A^{N-1}B]$ has full row rank [3]. However, the Kalman rank condition is not necessarily useful for model reduction because it cannot capture the controllability of systems quantitatively. Such a quantitative characterization of controllability plays an important role in performing an approximation error analysis.

Let us seek another characterization of controllability that has good compatibility with an error analysis for clustered model reduction. To this end, we first provide the following lemma that gives a particular realization of Σ , called the controller-Hessenberg form:

Lemma 1. For any a second-order network Σ in (6), there exists a unitary matrix $H \in \mathbb{R}^{N \times N}$ such that $\mathfrak{A} := H^\top AH \in \mathbb{R}^{N \times N}$ and $\mathfrak{B} := H^\top B \in \mathbb{R}^N$ are in the form of

$$\mathfrak{A} = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \cdots & \alpha_{1,N} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \cdots & \alpha_{2,N} \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha_{N,N-1} & \alpha_{N,N} \end{bmatrix}, \quad \mathfrak{B} = \begin{bmatrix} \beta_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (7)$$

Furthermore, the dimension of the controllable subspace of Σ is given by

$$\nu := \begin{cases} \min_{i \in \{1, \dots, N-1\}} \{i : \alpha_{i+1,i} = 0\}, & \text{if } \prod_{i=1}^{N-1} \alpha_{i+1,i} = 0 \\ N, & \text{otherwise.} \end{cases} \quad (8)$$

Proof. As shown in Chapter 11.2.2 of [3], the application of the Arnoldi procedure to the pair (A, B) yields the controller-Hessenberg form $(\mathfrak{A}, \mathfrak{B})$. To show that the dimension of the controllable subspace is equal to ν in (8), let us consider the partition of $(\mathfrak{A}, \mathfrak{B})$ as

$$\mathfrak{A} = \begin{bmatrix} \mathfrak{A}_1 & \mathfrak{A}_{1,2} \\ 0 & \mathfrak{A}_{2,2} \end{bmatrix}, \quad \mathfrak{B} = \begin{bmatrix} \mathfrak{B}_1 \\ 0 \end{bmatrix}.$$

where $\mathfrak{A}_1 \in \mathbb{R}^{\nu \times \nu}$ and $\mathfrak{B}_1 \in \mathbb{R}^\nu$. From the block structure of $(\mathfrak{A}, \mathfrak{B})$, it is readily follows that

$$\text{rank}[\mathfrak{B}, \mathfrak{A}\mathfrak{B}, \dots, \mathfrak{A}^{N-1}\mathfrak{B}] = \text{rank}[\mathfrak{B}_1, \mathfrak{A}_{1,1}\mathfrak{B}_1, \dots, \mathfrak{A}_{1,1}^{\nu-1}\mathfrak{B}_1] \leq \nu.$$

Thus, it suffices to show that the pair $(\mathfrak{A}_{1,1}, \mathfrak{B}_1)$ is controllable. From the properties of the Arnoldi procedure shown in Chapter 10.4.5 of [3], we see that the first ν columns of H , denoted by $H_1 \in \mathbb{R}^{N \times \nu}$, span the controllable subspace of (A, B) , namely

$$\text{im } H_1 = \text{im} [B, AB, \dots, A^{N-1}B] = \text{im} [B, AB, \dots, A^{\nu-1}B]$$

where the second equality stems from the A -invariance of the controllable subspace. From the fact that $H_1 H_1^\top \in \mathbb{R}^{N \times N}$ is the orthogonal projection matrix onto $\text{im } H_1$, it follows that

$$H_1 H_1^\top A^{k-1} B = A^{k-1} B, \quad \forall k \in \{1, \dots, \nu\}.$$

Note that $\mathfrak{A}_{1,1} = H_1^\top A H_1$ and $\mathfrak{B}_1 = H_1^\top B$. Thus, we have

$$\text{rank}[\mathfrak{B}_1, \mathfrak{A}_{1,1}\mathfrak{B}_1, \dots, \mathfrak{A}_{1,1}^{\nu-1}\mathfrak{B}_1] = \text{rank } H_1^\top [B, AB, \dots, A^{\nu-1}B] = \nu,$$

which proves the controllability of $(\mathfrak{A}_{1,1}, \mathfrak{B}_1)$. \square

Note that the controller-Hessenberg form of Σ in Lemma 1 has the serially cascaded structure as shown in (7). From this particular structure, it follows that Σ is controllable if and only if $\alpha_{i+1,i} \neq 0$ for all $i \in \{1, \dots, N-1\}$. Based on the controller-Hessenberg form of Σ , we derive a frequency domain characterization of controllability that has good compatibility with model reduction as follows:

Lemma 2. Given a second-order network Σ in (6), consider \mathfrak{A} and \mathfrak{B} with H shown in Lemma 1. Define

$$\Phi := H \text{diag}(\gamma_1, \dots, \gamma_N) \in \mathbb{R}^{N \times N} \quad (9)$$

where

$$\gamma_i := \|(e_i^N)^\top (sI_N - \mathfrak{A})^{-1} \mathfrak{B}\|_{\mathcal{H}_\infty}.$$

Then, Σ is controllable if and only if Φ is nonsingular.

Proof. Note that the uncontrollability of Σ is equivalent to the existence of a nonzero vector $\eta \in \mathbb{R}^N$ such that

$$\eta^\top (sI_N - A)^{-1} B = 0. \quad (10)$$

Thus, to prove the claim, it suffices to show that (10) is equivalent to

$$\eta^\top \Phi = 0. \quad (11)$$

Let

$$\mathfrak{X}(s) := (sI_{2n} - \mathfrak{A})^{-1} \mathfrak{B}, \quad (12)$$

whose i th element is denoted by \mathfrak{X}_i . Consider the dimension of the controllable subspace of Σ , denoted by ν in (8), and let

$$\mathcal{J} := \{1, \dots, \nu\}, \quad \overline{\mathcal{J}} := \{\nu + 1, \dots, N\}.$$

Since $\mathfrak{X}_i(s) \equiv 0$ for $i \in \overline{\mathcal{J}}$, we have

$$\mathfrak{X}(s) = e_{\mathcal{J}}^N (e_{\mathcal{J}}^N)^\top \mathfrak{X}(s). \quad (13)$$

Noting that

$$(sI_{2n} - A)^{-1} B = H \mathfrak{X}(s),$$

we see that (10) is equivalent to

$$\eta^\top H e_{\mathcal{J}}^N (e_{\mathcal{J}}^N)^\top \mathfrak{X}(s) = 0. \quad (14)$$

The functions $\mathfrak{X}_i(s)$ for $i \in \mathcal{J}$ are linearly independent because any two of them do not have the same relative degree, i.e., the difference of degrees between the denominator and the numerator polynomials, due to the serially cascaded structure in (7). This fact implies that (14) is equivalent to $\eta^\top H e_{\mathcal{J}}^N = 0$. On the other hand, from $\gamma_i = \|\mathfrak{X}_i\|_{\mathcal{H}_\infty}$ leading to $\gamma_i = 0$ for $i \in \overline{\mathcal{J}}$, it follows that

$$\text{diag}(\gamma_1, \dots, \gamma_N) = e_{\mathcal{J}}^N (e_{\mathcal{J}}^N)^\top \text{diag}(\gamma_1, \dots, \gamma_N).$$

Thus, (11) is also equivalent to $\eta^\top H e_{\mathcal{J}}^N = 0$. Hence, the claim follows. \square

Lemma 2 provides the characterization of controllability in the frequency domain. The index matrix Φ in (9) is composed of the unitary transformation matrix H weighted by the maximal gains of the input-to-state mapping in the controller-Hessenberg form. Using this lemma, we can prove the following result on the exact clustered model reduction:

Theorem 1. Given a second-order network Σ in (6), consider a cluster set $\{\mathcal{I}_{[l]}\}_{l \in \mathbb{L}}$. For each cluster $\mathcal{I}_{[l]}$, if there exist $p_{[l]} \in \mathbb{R}^{1 \times |\mathcal{I}_{[l]|}$, $\phi_{1[l]}^* \in \mathbb{R}^{1 \times 2n}$ and $\phi_{2[l]}^* \in \mathbb{R}^{1 \times 2n}$ such that $\|p_{[l]}\| = 1$ and

$$(e_{\mathcal{I}_{[l]}}^n)^\top \Phi_1 = p_{[l]}^\top \phi_{1[l]}^*, \quad (e_{\mathcal{I}_{[l]}}^n)^\top \Phi_2 = p_{[l]}^\top \phi_{2[l]}^* \quad (15)$$

where $\Phi_1 \in \mathbb{R}^{n \times 2n}$ and $\Phi_2 \in \mathbb{R}^{n \times 2n}$ denote the upper and lower half components of Φ in (9), respectively, then the aggregated model $\hat{\Sigma}$ in (3) given by P in (2) satisfies

$$g(s) = \hat{g}(s), \quad (16)$$

where g and \hat{g} are defined as in (5). Furthermore, if (15) holds, then it follows that

$$p_{[l]} = \left(\frac{(e_{\mathcal{I}_{[l]}}^n)^\top K^{-1} f}{\|(e_{\mathcal{I}_{[l]}}^n)^\top K^{-1} f\|} \right)^\top, \quad \forall l \in \mathbb{L}. \quad (17)$$

Proof. Using $\mathcal{P} := \text{diag}(P, P)$, we can verify that $\hat{\Sigma}$ in (3) is represented as the first-order realization

$$\hat{\Sigma} : \begin{cases} \dot{\Xi} = \mathcal{P}A\mathcal{P}^\top \Xi + \mathcal{P}Bu \\ \hat{x} = C\mathcal{P}^\top \Xi \end{cases}$$

where $\Xi := [\xi^\top, \dot{\xi}^\top]^\top \in \mathbb{R}^{2L}$. Note that (15) holds for each $\mathcal{I}_{[l]}$ if and only if there exists $\bar{p}_{[l]} \in \mathbb{R}^{(|\mathcal{I}_{[l]|-1}) \times |\mathcal{I}_{[l]|}$ such that $[p_{[l]}^\top, \bar{p}_{[l]}^\top]^\top$ is unitary and

$$\bar{\mathcal{P}}\Phi = \begin{bmatrix} \bar{\mathcal{P}}\Phi_1 \\ \bar{\mathcal{P}}\Phi_2 \end{bmatrix} = 0 \quad (18)$$

where

$$\bar{\mathcal{P}} := \text{diag}(\bar{P}, \bar{P}) \in \mathbb{R}^{2(n-L) \times 2n}, \quad \bar{P} := \text{diag}(\bar{p}_{[1]}, \dots, \bar{p}_{[L]})\Pi \in \mathbb{R}^{(n-L) \times n}. \quad (19)$$

From the construction of \mathcal{P} and $\bar{\mathcal{P}}$, we see that $[\mathcal{P}^\top, \bar{\mathcal{P}}^\top]^\top$ is unitary, i.e., $\mathcal{P}^\top \mathcal{P} + \bar{\mathcal{P}}^\top \bar{\mathcal{P}} = I_{2n}$. Thus, the similarity transformation of $g - \hat{g}$ by

$$V = \begin{bmatrix} \mathcal{P} & I_{2L} \\ I_{2n} & 0 \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} 0 & I_{2n} \\ I_{2L} & -\mathcal{P} \end{bmatrix}$$

yields

$$VA_e V^{-1} = \begin{bmatrix} \mathcal{P}A\mathcal{P}^\top & \mathcal{P}A\bar{\mathcal{P}}^\top \bar{\mathcal{P}} \\ 0 & A \end{bmatrix}, \quad VB_e = \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad C_e V^{-1} = \begin{bmatrix} C\mathcal{P}^\top & C\bar{\mathcal{P}}^\top \bar{\mathcal{P}} \end{bmatrix}$$

where

$$A_e := \begin{bmatrix} A & 0 \\ 0 & \mathcal{P}A\mathcal{P}^\top \end{bmatrix}, \quad B_e := \begin{bmatrix} V \\ -\mathcal{P}B \end{bmatrix}, \quad C_e := \begin{bmatrix} C & C\mathcal{P}^\top \end{bmatrix}.$$

This block structure implies that the error system admits the factorization of

$$g(s) - \hat{g}(s) = \Theta(s) \bar{\mathcal{P}}^\top \bar{\mathcal{P}} (sI_{2n} - A)^{-1} B, \quad \Theta(s) := C\mathcal{P}^\top (sI_{2L} - \mathcal{P}A\mathcal{P}^\top)^{-1} \mathcal{P}A + C. \quad (20)$$

Note that (18) is equivalent to

$$\bar{\mathcal{P}}(sI_{2n} - A)^{-1} B = 0, \quad (21)$$

which can be confirmed by the equivalence between (10) and (11). Hence, (16) is verified. Finally, substituting $s = 0$ to (21), we have $-\bar{\mathcal{P}}A^{-1}B = 0$, which can be rewritten as

$$\begin{bmatrix} \bar{P} & 0 \\ 0 & \bar{P} \end{bmatrix} \begin{bmatrix} K^{-1} f \\ 0 \end{bmatrix} = 0.$$

Thus, $\bar{p}_{[l]}(e_{\mathcal{I}_{[l]}}^n)^\top K^{-1} f = 0$ holds for any $l \in \mathbb{L}$. This proves that $p_{[l]}$ is given by (17). \square

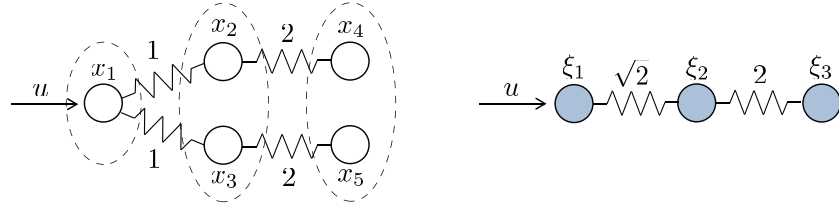


Fig. 1. Exact clustered model reduction of a simple second-order network.

As shown in Theorem 1, the aggregation of clusters satisfying (15) causes no approximation error. This exact dimension reduction is based on the elimination of local uncontrollable subspace characterized by (15). An intuitive interpretation of this exact clustered model reduction is explained through the following example:

Example. Let us consider a second-order network Σ in (1) given by

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 3 & -1 & -1 & 0 & 0 \\ -1 & 3 & 0 & -2 & 0 \\ -1 & 0 & 3 & 0 & -2 \\ 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & 2 \end{bmatrix}, \quad f = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

whose interconnection topology is depicted in the left of Fig. 1, where the i th element of x is denoted by x_i . The symmetric topology with respect to permutation suggests that the trajectories of x_2 and x_3 , as well as those of x_4 and x_5 are identical if

$$x_2(0) = x_3(0), \quad \dot{x}_2(0) = \dot{x}_3(0), \quad x_4(0) = x_5(0), \quad \dot{x}_4(0) = \dot{x}_5(0),$$

or equivalently

$$g_2(s) = g_3(s), \quad g_4(s) = g_5(s)$$

where g_i denotes the i th element of g in (5). Thus, the subspaces x_2-x_3 and x_4-x_5 are uncontrollable, i.e., x_2 and x_3 as well as x_4 and x_5 can be exactly aggregated for reduction. In the following, we characterize this local uncontrollability by using Theorem 1.

By the controller-Hessenberg transformation of Σ , we obtain \mathfrak{A} and \mathfrak{B} in (7) with its transformation matrix H , which lead to the index matrix

$$\Phi = \begin{bmatrix} 0 & 1.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.000 & 0 & 0 & 0 & 0 \\ -0.768 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.234 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.234 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.226 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.226 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the second and third row vectors, the fourth and fifth row vectors, the seventh and eighth row vectors, and the ninth and tenth row vectors are identical, respectively. This implies that, for each of the clusters

$$\mathcal{I}_{[1]} = \{1\}, \quad \mathcal{I}_{[2]} = \{2, 3\}, \quad \mathcal{I}_{[3]} = \{4, 5\},$$

which are depicted by the chain circles in Fig. 1, there exist $\phi_{1[l]}^* \in \mathbb{R}^{1 \times 10}$ and $\phi_{2[l]}^* \in \mathbb{R}^{1 \times 10}$ such that (15) holds with

$$p_{[1]} = 1, \quad p_{[2]} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad p_{[3]} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

which indeed comply with (17) since $K^{-1}f \in \mathbb{R}^5$ coincides with the all-ones vector. Thus, the aggregation matrix P in (2) is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Accordingly, we obtain the aggregated model $\hat{\Sigma}$ in (3) with

$$PDP^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad PKP^T = \begin{bmatrix} 3 & -\sqrt{2} & 0 \\ -\sqrt{2} & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}, \quad Pf = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

which satisfies (16). The interconnection topology of this aggregated model is depicted in the right of Fig. 1, where the i th element of ξ is denoted by ξ_i .

2.2.2 Approximation Error Analysis in terms of the \mathcal{H}_∞ -Norm

In Section 2.2.1, we have developed the exact clustered model reduction method by aggregating clusters such that (15). Even though such an exact approximation is indeed desirable, the reduction of system dimensions should be restrictive. Thus, in the following, we aim at enhancing this result to that with a small approximation error. To perform an approximation error analysis in terms of the \mathcal{H}_∞ -norm, we define the following notion of cluster reducibility, which is defined as weak controllability of local subsystem states:

Definition 2. Let a second-order network Σ in (6) be given. A cluster $\mathcal{I}_{[l]}$ is said to be θ -reducible if there exist $\phi_{1[l]}^* \in \mathbb{R}^{1 \times 2n}$ and $\phi_{2[l]}^* \in \mathbb{R}^{1 \times 2n}$ such that

$$\max \left\{ \left\| (e_{\mathcal{I}_{[l]}}^n)^T \Phi_1 - p_{[l]}^T \phi_{1[l]}^* \right\|_{l_\infty}, \left\| (e_{\mathcal{I}_{[l]}}^n)^T \Phi_2 - p_{[l]}^T \phi_{2[l]}^* \right\|_{l_\infty} \right\} \leq \theta, \quad \theta \geq 0 \quad (22)$$

where $\Phi_1 \in \mathbb{R}^{n \times 2n}$ and $\Phi_2 \in \mathbb{R}^{n \times 2n}$ denote the upper and lower half components of Φ in (9), respectively, and $p_{[l]}$ is defined as in (17).

In Definition 2, the constant θ represents a quantitative index of cluster reducibility, i.e., a quantitative index for the controllability of local subsystem states. Obviously, since (17) holds, the relaxed condition in (22) includes the exact condition in (15), and it is equivalent to (15) if $\theta = 0$. By the aggregation of θ -reducible clusters, we obtain the following aggregated model satisfying an \mathcal{H}_∞ -error bound that has a linear relation with the value of θ :

Theorem 2. Let a second-order network Σ in (6) be given, and define an aggregation matrix P in (2) with (17). If all clusters are θ -reducible, then the aggregated model $\hat{\Sigma}_P$ in (3) is stable and satisfies

$$g(0) = \hat{g}(0), \quad \|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty} \leq \sigma \sqrt{2 \sum_{l=1}^L |\mathcal{I}_{[l]}| (|\mathcal{I}_{[l]}| - 1) \theta} \quad (23)$$

where g and \hat{g} are defined as in (5), and

$$\sigma := \left\| P^T (s^2 I_L + s P D P^T + P K P^T)^{-1} [P K \ P D] - [I_n \ 0] \right\|_{\mathcal{H}_\infty}.$$

Proof. Using the factorization of (20), we have

$$\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty} \leq \|\Theta(s) \bar{P}^T\|_{\mathcal{H}_\infty} \|\bar{P}(s I_{2n} - A)^{-1} B\|_{\mathcal{H}_\infty}.$$

Note that

$$\mathcal{P}A\bar{\mathcal{P}}^\top = \begin{bmatrix} 0 & P\bar{\mathcal{P}}^\top \\ -PK\bar{\mathcal{P}}^\top & -PD\bar{\mathcal{P}}^\top \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -PK & -PD \end{bmatrix} \bar{\mathcal{P}}^\top.$$

Thus, it follows that

$$\|\Theta(s)\bar{\mathcal{P}}^\top\|_{\mathcal{H}_\infty} = \left\| \{-P^\top(s^2I_L + sPDP^\top + PKP^\top)^{-1}[PK \ PD] + C\} \bar{\mathcal{P}}^\top \right\|_{\mathcal{H}_\infty} = \sigma.$$

In the following, we use the notation of

$$\bar{\mathcal{P}}(sI_{2n} - A)^{-1}B = \begin{bmatrix} \bar{P}G_1(s) \\ \bar{P}G_2(s) \end{bmatrix}, \quad \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} := (sI_{2n} - A)^{-1}B.$$

In this notation, let us evaluate

$$\|\bar{\mathcal{P}}(sI_{2n} - A)^{-1}B\|_{\mathcal{H}_\infty} \leq \sqrt{\sum_{l=1}^L \left\| \bar{p}_{[l]}(e_{\mathcal{I}_{[l]}}^n)^\top G_1(s) \right\|_{\mathcal{H}_\infty}^2 + \sum_{l=1}^L \left\| \bar{p}_{[l]}(e_{\mathcal{I}_{[l]}}^n)^\top G_2(s) \right\|_{\mathcal{H}_\infty}^2}$$

To this end, we first prove that

$$\|CG_i(s)\|_{\mathcal{H}_\infty} \leq \sqrt{p} \|C\Phi_i\|_{l_\infty}, \quad i \in \{1, 2\} \quad (24)$$

for any $C \in \mathbb{R}^{p \times n}$. Note that

$$\|CG_i(s)\|_{\mathcal{H}_\infty} \leq \sqrt{\sum_{k=1}^p \|C_k G_i(s)\|_{\mathcal{H}_\infty}^2} \leq \sqrt{p} \max_k \|C_k G_i(s)\|_{\mathcal{H}_\infty}, \quad i \in \{1, 2\}$$

where $C_k \in \mathbb{R}^{1 \times n}$ denotes the k th row of C . Furthermore, it follows that

$$C_k G_1(s) = \sum_{i=1}^n C_{k,i} \sum_{j=1}^n H_{i,j} \mathfrak{X}_j(s), \quad C_k G_2(s) = \sum_{i=1}^n C_{k,i} \sum_{j=n+1}^{2n} H_{i,j} \mathfrak{X}_j(s)$$

where $H_{i,j}$ and $C_{i,j}$ denote the (i, j) -elements of H and C , respectively, and \mathfrak{X}_j denotes the j th element of \mathfrak{X} defined as in (12). Therefore, we have

$$\|C_k G_1(s)\|_{\mathcal{H}_\infty} = \left\| \sum_{i=1}^n C_{k,i} \sum_{j=1}^n H_{i,j} \mathfrak{X}_j(s) \right\|_{\mathcal{H}_\infty} \leq \sum_{j=1}^n \left| \sum_{i=1}^n C_{k,i} H_{i,j} \right| \gamma_j = \|C_k \Phi_1\|_{l_\infty},$$

where $\gamma_j = \|\mathfrak{X}_j\|_{\mathcal{H}_\infty}$ has been used. Similarly, we have $\|C_k G_2\|_{\mathcal{H}_\infty} \leq \|C_k \Phi_2\|_{l_\infty}$. By the definition of the l_∞ -induced norm, it follows that

$$\max_k \|C_k \Phi_i\|_{l_\infty} = \|C \Phi_i\|_{l_\infty}, \quad i \in \{1, 2\}.$$

Hence, (24) is verified. Using (24) with $C = \bar{p}_{[l]}(e_{\mathcal{I}_{[l]}}^n)^\top \in \mathbb{R}^{(|\mathcal{I}_{[l]}|-1) \times n}$, we obtain

$$\left\| \bar{p}_{[l]}(e_{\mathcal{I}_{[l]}}^n)^\top G_i(s) \right\|_{\mathcal{H}_\infty} \leq \sqrt{|\mathcal{I}_{[l]}| - 1} \left\| \bar{p}_{[l]}(e_{\mathcal{I}_{[l]}}^n)^\top \Phi_i \right\|_{l_\infty}, \quad i \in \{1, 2\}.$$

The θ -reducibility of all clusters ensures that

$$\|\Delta_{i[l]}\|_{l_\infty} \leq \theta, \quad \Delta_{i[l]} := (e_{\mathcal{I}_{[l]}}^n)^\top \Phi_i - p_{[l]}^\top \phi_{i[l]}^*, \quad i \in \{1, 2\}.$$

Thus, we have

$$\left\| \bar{p}_{[l]}(e_{\mathcal{I}_{[l]}}^n)^\top \Phi_i \right\|_{l_\infty} = \|\bar{p}_{[l]} \Delta_{i[l]}\|_{l_\infty} \leq \|\bar{p}_{[l]}\|_{l_\infty} \|\Delta_{i[l]}\|_{l_\infty} \leq \sqrt{|\mathcal{I}_{[l]}|} \theta, \quad i \in \{1, 2\},$$

which leads to

$$\|\bar{\mathcal{P}}(sI_{2n} - A)^{-1}B\|_{\mathcal{H}_\infty} \leq \sqrt{2 \sum_{l=1}^L |\mathcal{I}_{[l]}| (|\mathcal{I}_{[l]}| - 1)} \theta.$$

This proves the bound in (23). Finally, $g(0) = \hat{g}(0)$ is proven by (17), which implies $\bar{\mathcal{P}}(sI_{2n} - A)^{-1}B = 0$ for $s = 0$. \square

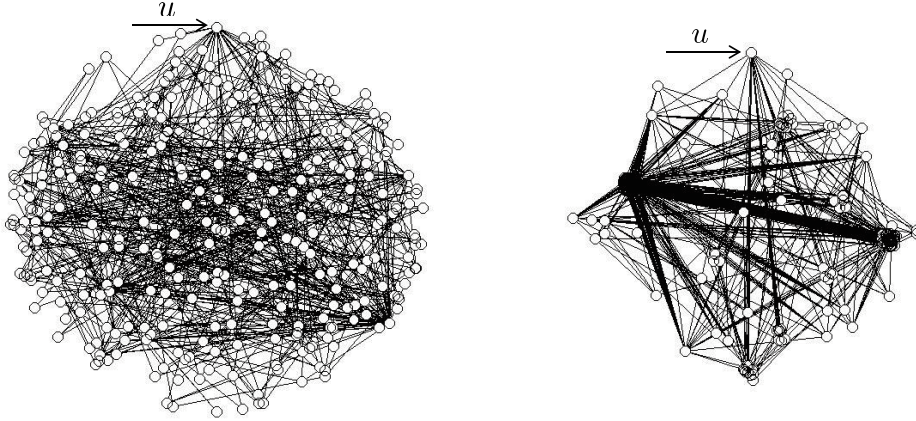


Fig. 2. Interconnection topologies of the original second-order network (300 nodes) and its aggregated model (42 clusters).

Theorem 2 shows a linear relation between the approximation error and the value of θ . Note that θ represents the degree of cluster reducibility, i.e., it captures the weak controllability of local subsystem states in a quantitative manner. By this theorem, we can regulate the approximation quality of the resultant aggregated models using θ as a design parameter.

3. Application to Complex Networks

In this section, we demonstrate the efficiency of Theorem 2 through an example of complex networks. We deal with a second-order network Σ in (1) evolving over the Holme-Kim model [1] composed of 300 nodes and about 600 edges, whose interconnection topology is depicted in the left of Fig. 2. This network is an extension of the Barabasi-Albert model, which is one of the best-known complex network models, and has a scale-free and small-world property, i.e., most nodes are not directly interconnected with the other nodes while most nodes have short-length paths from every other node, as well as a high cluster coefficient. As complying with the Holme-Kim model, we specify $K \in \mathbb{R}^{300 \times 300}$ as

$$K_{i,j} = \begin{cases} -1, & \text{if the } i\text{th and } j\text{th nodes are connected} \\ 0, & \text{otherwise,} \end{cases} \quad K_{i,i} = \begin{cases} 1 - \sum_{j=1, j \neq i}^{500} k_{1,j}, & i = 1 \\ - \sum_{j=1, j \neq i}^{500} k_{i,j}, & i \neq 1, \end{cases}$$

where $K_{i,j}$ denotes the (i,j) -element of K . This construction implies that only the first mass component is subject to the fixed boundary condition while the others are subject to free boundary conditions. Furthermore, we set the other system parameters as $D = 0.1 \times I_{300}$ and $f = e_1^{300}$.

Against each value of θ , in the left of Fig. 3, we plot the resultant number of nodes of the aggregated model. Furthermore, in the right of Fig. 3, we plot the resultant relative approximation error. These figures show that, as θ decreases, the dimension of the aggregated model increases monotonically, and the approximation error decreases almost monotonically. This result confirms that the value of θ successfully captures the approximation quality of the resultant aggregated models. By decreasing the value of θ , we find that, when $\theta = 1.762$, the original 600-dimensional second-order network is reduced to an 84-dimensional version, and the relative approximation error turns out to be

$$\frac{\|g(s) - \hat{g}(s)\|_{\mathcal{H}_\infty}}{\|g(s)\|_{\mathcal{H}_\infty}} = 0.0481.$$

The interconnection topology of the aggregated model is depicted in the right of Fig. 2, where a set of edges that are essential to the input-to-state mapping is highlighted.

Finally, we show the impulse response of the original second-order network and the 84-dimensional and 4-dimensional aggregated models. In Fig. 4, we plot the trajectories of x_1 and x_2 of the original system along with those of the aggregated models. From this figure, we can see that the behavior of

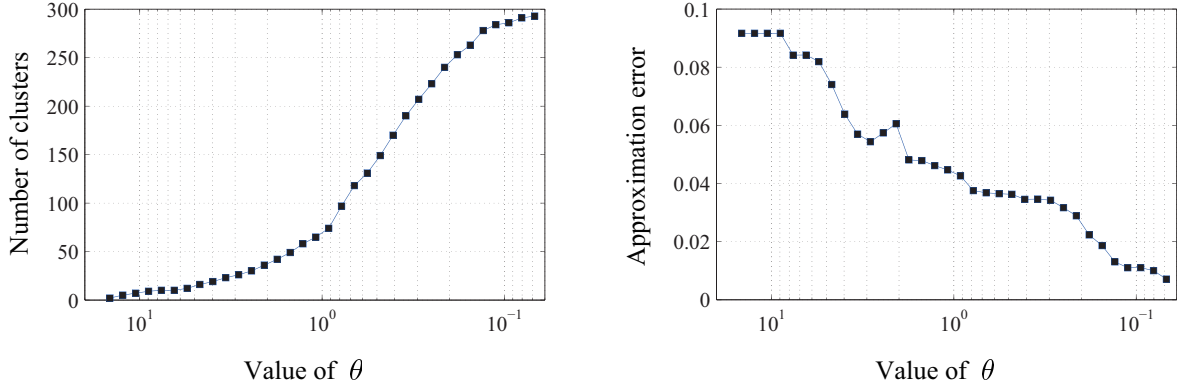


Fig. 3. Resultant number of clusters and relative approximation error versus the value of θ .

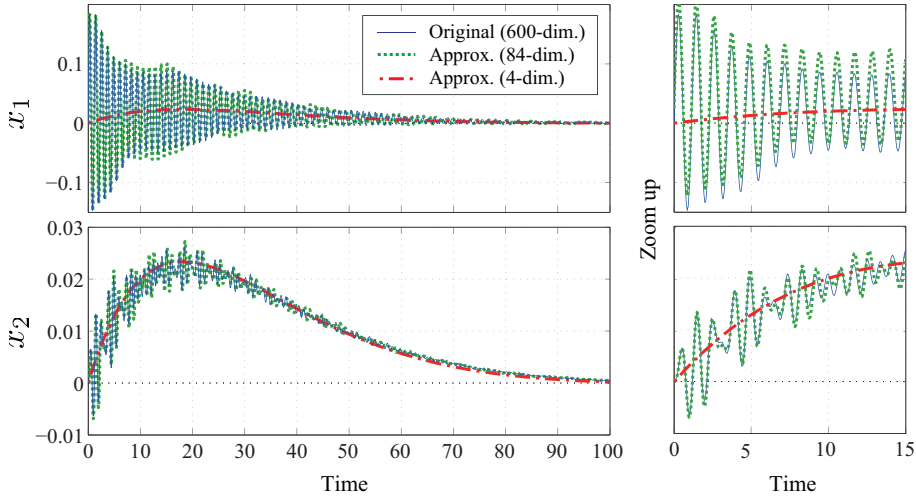


Fig. 4. Impulse responses of the first and second mass components.

the original states is well approximated by that of the 84-dimensional aggregated model, while the 4-dimensional version captures only the rough behavior of the original states. From this example, we can conclude that the proposed clustered model reduction method successfully extracts meaningful cluster interconnections from a viewpoint of the input-to-state mapping approximation.

4. Conclusions

In this paper, we have developed a clustered model reduction method for interconnected second-order systems evolving over undirected networks. In this method, network clustering, i.e., clustering of subsystems, is performed according to a kind of weak controllability of local subsystem states, which we call cluster reducibility. It has been found that the cluster reducibility is characterized for second-order networks based on the controller-Hessenberg transformation of their first-order representation, and the aggregation of the reducible clusters yields an aggregated model that preserves a network structure among clustered subsystems. Furthermore, we have shown that the resultant aggregated model satisfies an \mathcal{H}_∞ -error bound of the state discrepancy caused by the cluster aggregation. Finally, the efficiency of the proposed method has been demonstrated by an example of complex networks.

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