# Dissipativity-Preserving Model Reduction Based on Generalized Singular Perturbation

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***Abstract*— In this paper, we develop a dissipativity-preserving model reduction method based on a generalized singular perturbation approximation. This model reduction framework can deal with not only standard singular perturbation approximation but also projection-based model reduction as a special case. To develop such a model reduction method, we investigate a condition under which system dissipativity is appropriately preserved through the approximation. Moreover, deriving a novel factorization of the error system in the Laplace domain, we derive an a priori error bound in terms of the $\mathcal{H}_2$-norm. The efficiency of the model reduction is shown through an example of interconnected second-order systems.**

## I. Introduction

Along with the recent dramatic developments in engineering, the architecture of systems that interest the control community has tended to become more complex and larger in scale [1]. In view of this, it is crucial to develop approximation methods that enables us to reduce the complexity of systems. Additionally, it is desirable that some particular structures of systems such as stability, dissipativity, and positivity are preserved through out the approximation. It is expected that this kind of structure-preserving model reduction has the potential to significantly simplify the analysis and synthesis of large-scale complex systems.

Against such a background, this paper addresses a model reduction problem that is formulated based on a generalized singular perturbation approximation. It is found that the model reduction framework based on a generalized singular perturbation approximation can deal with not only the standard singular perturbation approximation but also the projection-based model reduction as a special case; see Section II for details. In this sense, this model reduction provides a unified framework for many model reduction methods.

In addition, we consider the preservation of system dissipativity. To this end, we first derive a tractable representation of reduced models, which provides a clear insight into achieving dissipativity preservation. In addition, deriving a novel factorization of the error system in the Laplace domain, we show that our generalized singular perturbation approximation admits an a priori error bound in terms of the $\mathcal{H}_2$-norm.

To clarify our contribution, some references for structure-preserving model reduction are in order. For example, [2] and [3] each address a model reduction problem while preserving a particular system structure such as the Lagrangian structure or the second-order structure. In particular, [4] and [5] develop model reduction methods with passivity preservation. However, these model reduction problems are not formulated on the premise of dissipativity, which corresponds to a generalized notion of passivity. Moreover, no global error bound has been derived. It should be finally noted that this paper provides a generalization of the results derived in [6] by the authors, where a passivity-preserving model reduction method based on the standard singular perturbation approximation has been developed.

This paper is organized as follows. First, in Section II, we formulate a dissipativity-preserving model reduction problem based on a notion of generalized singular perturbation approximations. It will be found that the generalized singular perturbation approximation can deal with not only the standard singular perturbation approximation but also the projection-based model reduction as a special case. Next, in Section III, we describe the main results of this paper, which include the derivation of a condition for dissipativity preservation and an a priori error bound in terms of the $\mathcal{H}_2$-norm. Then, in Section III-C, we demonstrate the efficiency of our model reduction method through an example of mass-spring-damper systems. Finally, concluding remarks are provided in Section IV.

**Notation** The following notation is to be used: $\mathbb{R}$: set of real numbers; $I_n$: unit matrix of size $n \times n$; $M \prec O_n$ ($M \preceq O_n$): negative (semi)definiteness of a symmetric matrix $M \in \mathbb{R}^{n\times n}$; $M \succ O_n$ ($M \succeq O_n$): positive (semi)definiteness of a symmetric matrix $M \in \mathbb{R}^{n\times n}$; $\mathrm{im}(M)$: range space spanned by the column vectors of a matrix $M$; $\mathrm{tr}(M)$: trace of a matrix $M$; $\mathrm{diag}(M_1, \ldots, M_n)$: block diagonal matrix having matrices $M_1, \ldots, M_n$ on its block diagonal.

The $\mathcal{H}_\infty$-norm of a stable proper transfer matrix $G$ and the $\mathcal{H}_2$-norm of a stable strictly proper transfer matrix $G$ are respectively defined by

$$\|G(s)\|_{\mathcal{H}_\infty} := \sup_{\omega\in\mathbb{R}} \|G(j\omega)\|,$$
$$\|G(s)\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}(G(j\omega)G^{\mathsf{T}}(-j\omega))d\omega\right)^{\frac{1}{2}}$$

where $\|\cdot\|$ denotes the induced 2-norm.

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## II. Problem Formulation

### A. Generalized Singular Perturbation Approximation

We mathematically formulate a model reduction framework based on a notion of the generalized singular perturbation approximation [7], [8]. Let us consider a linear system

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m_u}$, $C \in \mathbb{R}^{m_y\times n}$ and $D \in \mathbb{R}^{m_y\times m_u}$. In much literature on the singular perturbation theory, it is assumed that system (1) is intrinsically decoupled into several subsystems having different time scales; see [9], [10]. Contrastingly, such an assumption is not made in this paper. Instead, by finding an appropriate coordinate transformation, we decouple system (1) into two subsystems in a general manner. Namely, we denote the set of projection matrices by

$$\mathcal{P}^{\hat{n}\times n} := \{P \in \mathbb{R}^{\hat{n}\times n} : PP^{\mathsf{T}} = I_{\hat{n}}, \quad \hat{n} \le n\}, \tag{2}$$

and we perform the coordinate transformation of $\Sigma$ with a unitary matrix $[P^{\mathsf{T}}, \overline{P}^{\mathsf{T}}]^{\mathsf{T}} \in \mathbb{R}^{n\times n}$ with $P \in \mathcal{P}^{\hat{n}\times n}$ and $\overline{P} \in \mathcal{P}^{(n-\hat{n})\times n}$. Then, we obtain

$$\tilde{\Sigma} : \begin{cases} \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} PAP^{\mathsf{T}} & PA\overline{P}^{\mathsf{T}} \\ \overline{P}AP^{\mathsf{T}} & \overline{P}A\overline{P}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} PB \\ \overline{P}B \end{bmatrix} u \\ y = \begin{bmatrix} CP^{\mathsf{T}} & C\overline{P}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + Du. \end{cases} \tag{3}$$

To reduce the dimension of $\tilde{\Sigma}$, we impose an *algebraic constraint* on the trajectory of $\eta$. More specifically, confining the dynamics of $\eta$ by $\dot{\eta} \equiv \sigma\eta$, we obtain

$$\sigma\hat{\eta} = \overline{P}AP^{\mathsf{T}}\hat{\xi} + \overline{P}A\overline{P}^{\mathsf{T}}\hat{\eta} + \overline{P}Bu, \tag{4}$$

where $\eta$ and $\xi$ are replaced with their approximants $\hat{\eta}$ and $\hat{\xi}$, respectively. As long as $\sigma I_{n-\hat{n}} - \overline{P}A\overline{P}^{\mathsf{T}}$ is nonsingular, the approximant $\hat{\eta}$ in (4) is obtained as

$$\begin{aligned} \hat{\eta} = (\sigma I_{n-\hat{n}} - \overline{P}A\overline{P}^{\mathsf{T}})^{-1}\overline{P}AP^{\mathsf{T}}\hat{\xi} \\ +(\sigma I_{n-\hat{n}} - \overline{P}A\overline{P}^{\mathsf{T}})^{-1}\overline{P}Bu. \end{aligned} \tag{5}$$

Substituting (5) into the equation with respect to $\dot{\xi}$, we have the generalized singular perturbation model

$$\hat{\Sigma}_\sigma : \begin{cases} \dot{\hat{\xi}} = \hat{A}\hat{\xi} + \hat{B}u \\ \hat{y} = \hat{C}\hat{\xi} + \hat{D}u \end{cases} \tag{6}$$

where

$$\begin{aligned} \hat{A} &:= PAP^{\mathsf{T}} + PA\Pi AP^{\mathsf{T}} \in \mathbb{R}^{\hat{n}\times\hat{n}} \\ \hat{B} &:= PB + PA\Pi B \in \mathbb{R}^{\hat{n}\times m_u} \\ \hat{C} &:= CP^{\mathsf{T}} + C\Pi AP^{\mathsf{T}} \in \mathbb{R}^{m_y\times\hat{n}} \\ \hat{D} &:= D + C\Pi B \in \mathbb{R}^{m_y\times m_u} \end{aligned} \tag{7}$$

and

$$\Pi := \overline{P}^{\mathsf{T}}(\sigma I_{n-\hat{n}} - \overline{P}A\overline{P}^{\mathsf{T}})^{-1}\overline{P} \in \mathbb{R}^{n\times n}. \tag{8}$$

Note that this $\Pi$ does not depend on the basis selection of the projection $\overline{P}$. This is because

$$\Pi = \overline{P}^{\mathsf{T}}H^{\mathsf{T}}(\sigma I_{n-\hat{n}} - H\overline{P}A\overline{P}^{\mathsf{T}}H^{\mathsf{T}})^{-1}H\overline{P}$$

holds for any unitary matrix $H \in \mathbb{R}^{(n-\hat{n})\times(n-\hat{n})}$. This implies that, for a fixed constant $\sigma \in \mathbb{R}$, the generalized singular perturbation model $\hat{\Sigma}_\sigma$ in (6) depends only on the choice of $P \in \mathcal{P}^{\hat{n}\times n}$.

### B. Dissipativity-Preserving Model Reduction Problem

To formulate a dissipativity-preserving model reduction problem, we begin with the following standard definition of (strict) dissipativity [11], [12].

*Definition 1:* A linear system $\Sigma$ in (1) is said to be $V$-*dissipative with respect to*

$$Q = Q^{\mathsf{T}} \in \mathbb{R}^{(m_u+m_y)\times(m_u+m_y)}$$

if there exists $V = V^{\mathsf{T}} \succ \mathcal{O}_n$ such that

$$\mathcal{F}_Q(A,B,C,D;V) \prec \mathcal{O}_{n+m_u} \tag{9}$$

holds for

$$\mathcal{F}_Q(A,B,C,D;V) := \begin{bmatrix} A^{\mathsf{T}}V + VA & VB \\ B^{\mathsf{T}}V & 0 \end{bmatrix} - \begin{bmatrix} C & D \\ 0 & I_{m_u} \end{bmatrix}^{\mathsf{T}} Q \begin{bmatrix} C & D \\ 0 & I_{m_u} \end{bmatrix}. \tag{10}$$

In linear systems theory, the inequality (9) is called a *dissipation inequality*, and the quadratic functions

$$f_V(x) := x^{\mathsf{T}}Vx \tag{11}$$

and

$$s_Q(y,u) := [\, y^{\mathsf{T}} \ u^{\mathsf{T}} \,]\, Q \begin{bmatrix} y \\ u \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{y,y} & Q_{y,u} \\ Q_{u,y} & Q_{u,u} \end{bmatrix} \tag{12}$$

are called *storage functions* and *supply functions*, respectively.

In the rest of this paper, we denote the transfer matrix of $\Sigma$ by

$$G(s) := C(sI_n - A)^{-1}B + D, \tag{13}$$

and the generalized singular perturbation approximant of $G$ associated with $P \in \mathcal{P}^{\hat{n}\times n}$ by

$$\hat{G}_\sigma(s;P) := \hat{C}(sI_{\hat{n}} - \hat{A})^{-1}\hat{B} + \hat{D}, \tag{14}$$

where $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ are defined as in (7). The aim of this paper is to provide a solution to the following dissipativity-preserving model reduction problem.

*Problem:* Consider a linear system $\Sigma$ in (1), and suppose that it is $V$-dissipative with respect to $Q$. Given a constant $\delta \ge 0$, find a generalized singular perturbation model $\hat{\Sigma}_\sigma$ in (6) such that it is $\hat{V}$-dissipative with respect to $Q$ and satisfies

$$\|G(s) - \hat{G}_\sigma(s;P)\|_{\mathcal{H}_2} \le \delta \tag{15}$$

where $G$ and $\hat{G}_\sigma$ are defined as in (13) and (14), respectively.

## III. Main Results

### A. Dissipativity Preservation

First of all, we derive a tractable condition under which the generalized singular perturbation approximation appropriately preserves system dissipativity in Definition 1. The following fact will be useful for arguments below.

*Lemma 1:* Let a linear system $\Sigma$ in (1) be given, and suppose that it is $V$-dissipative with respect to $Q$. Consider a Cholesky factor $V_{\frac{1}{2}}$ of $V$ such that $V = V_{\frac{1}{2}}^{\mathsf{T}} V_{\frac{1}{2}}$. Then

$$\mathcal{F}_Q(V_{\frac{1}{2}} A V_{\frac{1}{2}}^{-1}, V_{\frac{1}{2}} B, C V_{\frac{1}{2}}^{-1}, D; I_n) \prec \mathcal{O}_{n+m_u} \tag{16}$$

holds.

*Proof:* It is found that $\mathcal{F}_Q(A, B, C, D; V)$ in (10) is rewritten as

$$\tilde{V}^{\mathsf{T}} \mathcal{F}_Q(V_{\frac{1}{2}} A V_{\frac{1}{2}}^{-1}, V_{\frac{1}{2}} B, C V_{\frac{1}{2}}^{-1}, D; I_n) \tilde{V}$$

where $\tilde{V} := \mathrm{diag}(V_{\frac{1}{2}}, I_{m_u})$. Hence, the claim follows. ■

This lemma shows that any $V$-dissipative system can be transformed to a system that is $I_n$-dissipative with respect to the same supply function. Owing to this fact, without loss of generality, we can assume that any dissipative system is $I_n$-dissipative, i.e., it admits the quadratic function $x^{\mathsf{T}} x$ as its storage function.

In projection-based model reduction methods, such a particular realization is actually useful for achieving dissipativity preservation. This is because, for any $P \in \mathcal{P}^{\hat{n} \times n}$, $\mathcal{F}_Q(PAP^{\mathsf{T}}, PB, CP^{\mathsf{T}}, D; I_{\hat{n}})$ is negative definite if and only if

$$\tilde{P} \mathcal{F}_Q(A, B, C, D; I_n) \tilde{P}^{\mathsf{T}}, \ \tilde{P} := \mathrm{diag}(P, I_{m_u})$$

is negative definite. This implies that the reduced model is $I_{\hat{n}}$-dissipative with respect to $Q$ whenever the original system is $I_n$-dissipative with respect to $Q$. However, due to the complicated form of $\hat{\Sigma}_\sigma$ in (6), the same conclusion for the generalized singular perturbation approximation seems nontrivial. In view of this, we derive a tractable representation of $\hat{A}$, which provides a clear insight into achieving dissipativity preservation.

*Lemma 2:* For any $A \in \mathbb{R}^{n \times n}$, $P \in \mathcal{P}^{\hat{n} \times n}$ and $\sigma \in \mathbb{R}$, the system matrix $\hat{A} \in \mathbb{R}^{\hat{n} \times \hat{n}}$ in (7) admits the representation

$$\hat{A} = (P + PA\Pi) A (P + PA\Pi)^{\mathsf{T}} - \sigma PA\Pi (PA\Pi)^{\mathsf{T}}, \tag{17}$$

where $\Pi \in \mathbb{R}^{n \times n}$ is defined as in (8). Moreover, $P + PA\Pi$ has full row rank.

*Proof:* First, we prove that $P + PA\Pi$ has full row rank, namely

$$\mathrm{rank}(P + PA\Pi) = \hat{n} \tag{18}$$

holds. Note that

$$\mathrm{rank}((P + PA\Pi) P^{\mathsf{T}} P) = \mathrm{rank}(P) = \hat{n}. \tag{19}$$

holds. Here, if we assume $\mathrm{rank}(P + PA\Pi) < \hat{n}$, then

$$\begin{aligned} &\mathrm{rank}((P + PA\Pi) P^{\mathsf{T}} P) \\ &\qquad \leq \min(\mathrm{rank}(P + PA\Pi), \mathrm{rank}(P^{\mathsf{T}} P)) < \hat{n}. \end{aligned}$$

This contradicts (19), and consequently (18) follows.

Next, we prove (17). We first prove that

$$\hat{A} P = (P + PA\Pi) A - \sigma PA\Pi \overline{P}^{\mathsf{T}} \overline{P} \tag{20}$$

holds. To this end, it suffices to show that

$$\Delta := \hat{A} P - (P + PA\Pi) A + \sigma PA\Pi \overline{P}^{\mathsf{T}} \overline{P} = 0.$$

Using

$$\Pi (\sigma I_n - A) \overline{P}^{\mathsf{T}} = \overline{P}^{\mathsf{T}}, \tag{21}$$

we obtain

$$\begin{aligned} \Delta &= PA\{(I_n + \Pi A) P^{\mathsf{T}} P - (I_n + \Pi A)\} + \sigma PA\Pi \overline{P}^{\mathsf{T}} \overline{P} \\ &= -PA(I_n + \Pi A) \overline{P}^{\mathsf{T}} \overline{P} + \sigma PA\Pi \overline{P}^{\mathsf{T}} \overline{P} \\ &= -PA \overline{P}^{\mathsf{T}} \overline{P} - PA\Pi A \overline{P}^{\mathsf{T}} \overline{P} + \sigma PA\Pi \overline{P}^{\mathsf{T}} \overline{P} \\ &= -PA \overline{P}^{\mathsf{T}} \overline{P} + PA\Pi (\sigma I_n - A) \overline{P}^{\mathsf{T}} \overline{P} \\ &= 0. \end{aligned}$$

Hence, (20) follows.

Multiplying (20) by $P^{\mathsf{T}}$ from the right side, we obtain

$$\begin{aligned} \hat{A} &= (P + PA\Pi) A P^{\mathsf{T}} \\ &= (P + PA\Pi) A (P + PA\Pi)^{\mathsf{T}} - (P + PA\Pi) A (PA\Pi)^{\mathsf{T}}. \end{aligned}$$

Furthermore, noting that (21) and $\Pi = \Pi \overline{P}^{\mathsf{T}} \overline{P}$ hold, we obtain

$$\sigma PA\Pi (PA\Pi)^{\mathsf{T}} - (P + PA\Pi) A (PA\Pi)^{\mathsf{T}} = 0$$

which implies $(P + PA\Pi) A (PA\Pi)^{\mathsf{T}} = \sigma PA\Pi (PA\Pi)^{\mathsf{T}}$. Thus, (17) follows. ■

This lemma shows that $\hat{A}$ in $\hat{\Sigma}_\sigma$ admits a *projection-like* factorization as in (17). Based on this fact, we can derive the following result on dissipativity preservation.

*Theorem 1:* Let a linear system $\Sigma$ in (1) be given, and suppose that it is $I_n$-dissipative with respect to $Q$. If $\sigma \geq 0$ and $P \in \mathcal{P}^{\hat{n} \times n}$ satisfies

$$\mathrm{im}(C^{\mathsf{T}}) \subseteq \mathrm{im}(P^{\mathsf{T}}), \tag{22}$$

then the generalized singular perturbation model $\hat{\Sigma}_\sigma$ in (6) is $I_{\hat{n}}$-dissipative with respect to $Q$.

*Proof:* Owing to (22), it follows that $\hat{C} = C(P + PA\Pi)^{\mathsf{T}}$ and $\hat{D} = D$. Noting that $\hat{B} = (P + PA\Pi) B$ holds, we can verify that $\mathcal{F}_Q(\hat{A}, \hat{B}, \hat{C}, \hat{D}; I_{\hat{n}})$ is rewritten as

$$\tilde{P} \mathcal{F}_Q(A, B, C, D; I_n) \tilde{P}^{\mathsf{T}} - \mathrm{diag}(2\sigma PA\Pi (PA\Pi)^{\mathsf{T}}, 0)$$

where $\tilde{P} = \mathrm{diag}(P + PA\Pi, I_{m_u})$. Here

$$\tilde{P} \mathcal{F}_Q(A, B, C, D; I_n) \tilde{P}^{\mathsf{T}}$$

is negative definite while

$$-\mathrm{diag}(2\sigma PA\Pi (PA\Pi)^{\mathsf{T}}, 0)$$

is negative semidefinite owing to the assumption of $\sigma \geq 0$. Thus, the claim follows. ■

This theorem shows that if the original system is $I_n$-dissipative $\Sigma$ with respect to a supply function, then the generalized singular perturbation model $\hat{\Sigma}_\sigma$ is $I_{\hat{n}}$-dissipative with respect to the same supply function as long as $\sigma \geq 0$ and (22) hold. Note that (22) can easily be satisfied by adding the basis of $\mathrm{im}(C^\mathsf{T})$ to $\mathrm{im}(P^\mathsf{T})$.

### *B. Approximation Error Analysis*

In this subsection, we analyze the approximation error caused by the generalized singular perturbation approximation. In literature on the standard singular perturbation theory, most of error analyses are developed in the time domain by using the asymptotic analysis [10], [13], or based on the premise of the balanced realization [14], [8]. In contrast to this, we develop error analysis in the Laplace domain without relying on the balanced realization. First, we derive a novel representation of the error system as shown in the following theorem.

*Theorem 2:* Given a transfer matrix $G$ in (13) and $\sigma \in \mathbb{R}$, define the generalized singular perturbation approximant $\hat{G}_\sigma$ in (14) associated with $P \in \mathcal{P}^{\hat{n}\times n}$. Then

$$
\begin{aligned}
&G(s) - \hat{G}_\sigma(s;P) \\
&\quad = \begin{cases} \hat{\Xi}_\sigma(s;P)\overline{P}^\mathsf{T}\overline{P}X_\sigma(s) \\ (\hat{\Xi}_\sigma(s;P)A + C)\overline{P}^\mathsf{T}\overline{P}\sigma^{-1}X_\sigma(s), \ \text{if } \sigma \neq 0. \end{cases}
\end{aligned} \tag{23}
$$

holds, where

$$
\begin{aligned}
\hat{\Xi}_\sigma(s;P) &:= \hat{C}(sI_{\hat{n}} - \hat{A})^{-1}(P + PA\Pi) + C\Pi \\
X_\sigma(s) &:= (\sigma I_n - A)(sI_n - A)^{-1}B - B
\end{aligned} \tag{24}
$$

with $\hat{A}$ and $\hat{C}$ are defined as in (7).

*Proof:* Denote the error system by

$$
G(s) - \hat{G}_\sigma(s;P) = C_e(sI_{n+\hat{n}} - A_e)^{-1}B_e + D_e
$$

where $A_e = \mathrm{Diag}(\hat{A}, A),\ B_e = [\hat{B}^\mathsf{T}, B^\mathsf{T}]^\mathsf{T},\ C_e = [-\hat{C}, C]$ and $D_e = -\hat{D}+D$. Considering the similarity transformation of the error system with

$$
T = \begin{bmatrix} I_{\hat{n}} & -P \\ 0 & I_n \end{bmatrix}, \qquad T^{-1} = \begin{bmatrix} I_{\hat{n}} & P \\ 0 & I_n \end{bmatrix},
$$

we have

$$
\begin{aligned}
&TA_eT^{-1} = \begin{bmatrix} \hat{A} & \hat{A}P - PA \\ 0 & A \end{bmatrix},\ TB_e = \begin{bmatrix} PA\Pi\overline{P}^\mathsf{T}\overline{P}B \\ B \end{bmatrix} \\
&C_eT^{-1} = \begin{bmatrix} -\hat{C} & -\hat{C}P + C \end{bmatrix},\ D_e = -C\Pi B,
\end{aligned} \tag{25}
$$

where $I - P^\mathsf{T}P = \overline{P}^\mathsf{T}\overline{P}$ is used. Using (20), we have

$$
\hat{A}P - PA = -PA\Pi\overline{P}^\mathsf{T}\overline{P}(\sigma I_n - A).
$$

Furthermore, using (21), we obtain

$$
-\hat{C}P + C = C\Pi(\sigma I_n - A).
$$

Thus, the block structure of (25) implies that the error system is given by the first one in (23). In addition

$$
\Pi\overline{P}^\mathsf{T} = \sigma^{-1}(\overline{P}^\mathsf{T} + \Pi A\overline{P}^\mathsf{T})
$$

follows from (21) if $\sigma \neq 0$. Substituting this into (23), we obtain the second one in (23). ■

The error system factorization shown in Theorem 2, which can be applied even to unstable systems, provides a qualitative insight on error analysis. That is, from the *cascaded* form of (23), we expect that the resultant approximation error will be small if the norm of $\overline{P}X_\sigma$ is sufficiently small, and the norm of $\hat{\Xi}_\sigma\overline{P}^\mathsf{T}$ or $(\hat{\Xi}_\sigma A + C)\overline{P}^\mathsf{T}$ is bounded. Furthermore, it is worth noting that $\hat{\Xi}_\sigma$ in (24) coincides with the generalized singular perturbation approximant of

$$
\Xi(s) = C(sI_n - A)^{-1} \tag{26}
$$

associated with $P \in \mathcal{P}^{\hat{n}\times n}$.

Now, we are ready to state a main result of this section. Utilizing Theorem 2 in conjunction with Theorem 1, we establish the following theorem that gives a solution to the structure-preserving model reduction problem in Section II as follows.

*Theorem 3:* Let a linear system $\Sigma$ in (1) be given, and suppose that it is $I_n$-dissipative with respect to $Q$. Assume that $Q_{y,y} \preceq \mathcal{O}_{m_y}$ holds for (12). Given $\sigma \geq 0$, let $\gamma > 0$ such that

$$
\mathcal{O}_n \succ \begin{cases} A + A^\mathsf{T} + \gamma^{-1}(I_n + C^\mathsf{T}C), \ \text{if } \sigma = 0 \\ A + A^\mathsf{T} + \gamma^{-1}(AA^\mathsf{T} + C^\mathsf{T}C), \ \text{otherwise.} \end{cases} \tag{27}
$$

Furthermore, let $\mathcal{W} = \mathcal{W}^\mathsf{T} \succeq \mathcal{O}_n$ such that

$$
A\mathcal{W} + \mathcal{W}A^\mathsf{T} + BB^\mathsf{T} = 0 \tag{28}
$$

holds. If $P \in \mathcal{P}^{\hat{n}\times n}$ satisfies

$$
\mathrm{im}([B, C^\mathsf{T}]) \subseteq \mathrm{im}(P^\mathsf{T}) \tag{29}
$$

$$
\epsilon \geq \begin{cases} \sqrt{\mathrm{tr}(\Phi_\sigma) - \mathrm{tr}(P\Phi_\sigma P^\mathsf{T})}, \ \text{if } \sigma = 0 \\ \sigma^{-1}\sqrt{\mathrm{tr}(\Phi_\sigma) - \mathrm{tr}(P\Phi_\sigma P^\mathsf{T})}, \ \text{otherwise.} \end{cases} \tag{30}
$$

where

$$
\Phi_\sigma := (\sigma I_n - A)\mathcal{W}(\sigma I_n - A)^\mathsf{T} \in \mathbb{R}^{n\times n}, \tag{31}
$$

then the generalized singular perturbation model $\hat{\Sigma}_\sigma$ in (14) is $I_{\hat{n}}$-dissipative with respect to $Q$ and satisfies

$$
G(\sigma) = \hat{G}_\sigma(\sigma;P), \quad \|G(s) - \hat{G}_\sigma(s;P)\|_{\mathcal{H}_2} \leq \gamma\epsilon \tag{32}
$$

where $G$ and $\hat{G}_\sigma$ are defined as in (13) and (14), respectively.

*Proof:* If $\Sigma$ is $I_n$-dissipative $\Sigma$ with respect to $Q$, then $\hat{\Sigma}_\sigma$ is $I_{\hat{n}}$-dissipative with respect to $Q$, as shown in Theorem 1. Note that $\Sigma$ and $\hat{\Sigma}_\sigma$ are both stable because they are $I_n$- and $I_{\hat{n}}$-dissipative with respect to $Q$ satisfying $Q_{y,y} \preceq \mathcal{O}_{m_y}$.

Next we prove (32). Note that $C\Pi = 0$ follows from (29). From Theorem 2, we have

$$
\begin{aligned}
&\|G(s) - \hat{G}_\sigma(s;P)\|_{\mathcal{H}_2} \\
&\quad \leq \begin{cases} \|\hat{\Xi}_\sigma\overline{P}^\mathsf{T}\|_{\mathcal{H}_\infty}\|\overline{P}X_\sigma\|_{\mathcal{H}_2}, \ \text{if } \sigma = 0 \\ \|(\hat{\Xi}_\sigma A + C)\overline{P}^\mathsf{T}\|_{\mathcal{H}_\infty}\|\overline{P}\sigma^{-1}X_\sigma\|_{\mathcal{H}_2}, \ \text{otherwise} \end{cases}
\end{aligned}
$$

where $\hat{\Xi}_\sigma$ and $X_\sigma$ are defined as in (24). Note that (29) implies that the feedthrough term of $\overline{P}X_\sigma$ is equal to zero. Thus, from (30), we can ensure that $\|\overline{P}X_\sigma\|_{\mathcal{H}_2} \leq \epsilon$ if $\sigma = 0$

and $\|\overline{P}\sigma^{-1}X_\sigma\|_{\mathcal{H}_2} \leq \epsilon$ otherwise. In what follows, we prove that

$$\gamma > \begin{cases} \|\hat{\Xi}_\sigma(s;P)\overline{P}^\mathsf{T}\|_{\mathcal{H}_\infty}, \text{ if } \sigma = 0 \\ \|(\hat{\Xi}_\sigma(s;P)A + C)\overline{P}^\mathsf{T}\|_{\mathcal{H}_\infty}, \text{ otherwise} \end{cases} \quad (33)$$

follows from (29) and (27).

First, we consider the case if $\sigma = 0$. Note that there always exists some $\gamma > 0$ such that (27) because $A + A^\mathsf{T} \prec \mathcal{O}_n$ holds. Here, owing to (29), the feedthrough term of $\hat{\Xi}_\sigma$ is equal to zero. Thus, from the bounded real lemma, it follows that $\|\hat{\Xi}_\sigma\|_{\mathcal{H}_\infty} < \gamma$ holds if there exists $\hat{V} = \hat{V}^\mathsf{T} \succ \mathcal{O}_{\hat{n}}$ such that

$$\begin{aligned} &\hat{V}\hat{A} + \hat{A}^\mathsf{T}\hat{V} \\ &\quad + \gamma^{-1}\left\{\hat{V}(P + PA\Pi)(P + PA\Pi)^\mathsf{T}\hat{V} + \hat{C}^\mathsf{T}\hat{C}\right\} \prec \mathcal{O}_{\hat{n}}. \end{aligned} \quad (34)$$

By the fact that $\hat{C} = CP^\mathsf{T} = C(P + PA\Pi)^\mathsf{T}$ holds, the inequality (34) with the solution of $\hat{V} = I_{\hat{n}}$ is rewritten as

$$\begin{aligned} (P + PA\Pi)\{A + A^\mathsf{T} + \gamma^{-1}(I_n + C^\mathsf{T}C)\}(P + PA\Pi)^\mathsf{T} \\ -2\sigma PA\Pi(PA\Pi)^\mathsf{T} \prec \mathcal{O}_{\hat{n}}, \end{aligned}$$

whose negative semidefiniteness is ensured by (27) and $-2PA\Pi(PA\Pi)^\mathsf{T} \preceq \mathcal{O}_{\hat{n}}$. Hence

$$\|\hat{\Xi}_\sigma(s)\overline{P}^\mathsf{T}\|_{\mathcal{H}_\infty} \leq \|\hat{\Xi}_\sigma(s)\|_{\mathcal{H}_\infty} < \gamma \quad (35)$$

follows. From an argument similar to this, we can verify that (29) and (27) ensure

$$\|(\hat{\Xi}_\sigma(s;P)A + C)\overline{P}^\mathsf{T}\|_{\mathcal{H}_\infty} \leq \|\hat{\Xi}_\sigma(s;P)A\|_{\mathcal{H}_\infty} < \gamma \quad (36)$$

if $\sigma \neq 0$. Finally, $G(\sigma) = \hat{G}_\sigma(\sigma;P)$ is proven by $X_\sigma(\sigma) = 0$ in (23). ■

Theorem 3 shows in (32) that the generalized singular perturbation approximation admits an a priori error bound. Note that the value of $\gamma$ in (32) corresponds to an upper bound for the gain of the state-to-output mapping of the generalized singular perturbation model.

Furthermore, to find $P \in \mathcal{P}^{\hat{n}\times n}$ such that (22) and (29) hold for a prescribed $\epsilon$, we can use the following procedure: First, we find the set $\{(\lambda_i, v_i)\}_{i\in\{1,\ldots,n\}}$ of all eigenpairs of $\Phi_\sigma$ in (31), where it is assumed without loss of generality that $\lambda_i \geq \lambda_{i+1}$ and $\|v_i\| = 1$. Next, we find $m \in \{1,\ldots,n\}$ such that

$$\epsilon^2 \geq \begin{cases} \lambda_{m+1} + \cdots + \lambda_n, \text{ if } \sigma = 0 \\ \sigma^{-1}(\lambda_{m+1} + \cdots + \lambda_n), \text{ otherwise} \end{cases} \quad (37)$$

and construct $V_m = [v_1,\ldots,v_m] \in \mathcal{P}^{n\times m}$. Finally, by the Gram-Schmidt process, we derive $P \in \mathcal{P}^{\hat{n}\times n}$ such that $\mathrm{im}(P^\mathsf{T}) = \mathrm{im}([V_m, B, C^\mathsf{T}])$.

It is worth noting that in the generalized singular perturbation approximation the resultant approximation error is related to the sum of neglected eigenvalues of $\Phi_\sigma$ as shown in (37). The major significance of Theorem 3 is the theoretical revelation that $\epsilon$ (i.e., the threshold of neglected eigenvalues of $\Phi_\sigma$) can be used as a design parameter to regulate the approximating quality of resultant approximate models.

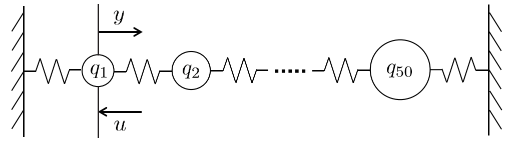


Fig. 1. Depiction of Mass-Spring-Damper System.

### C. Numerical Example

In this subsection, we demonstrate the efficiency of our generalized singular perturbation approximation through a numerical example. Let us consider the following mass-spring-damper system

$$\begin{cases} M\ddot{q} + R\dot{q} + Kq = Fu \\ y = Hq \end{cases} \quad (38)$$

where $M \succ \mathcal{O}_\nu$ denotes a mass matrix, $R \succ \mathcal{O}_\nu$ denotes a damper matrix, $K \succ \mathcal{O}_\nu$ denotes a spring stiffness matrix, $F \in \mathbb{R}^{\nu\times m_w}$ denotes a matrix describing actuator allocation, and $H \in \mathbb{R}^{m_z\times\nu}$ denotes a matrix describing sensor allocation. This second-order system is often used as a primary model of flexible mechanical systems in vibration suppression control [15] and the rotor dynamics in power system stabilization [16].

Let $x_0 := [q^\mathsf{T}, \dot{q}^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{2\nu}$ be the state variable of this system. Then, we have the $2\nu$-dimensional system $\Sigma$ in (1), with

$$A = \begin{bmatrix} 0 & I_\nu \\ -M^{-1}K & -M^{-1}R \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix}, \quad C = \begin{bmatrix} H & 0 \end{bmatrix}, \quad D = 0.$$

Let us consider a case in which $\nu = 50$ mass components are coupled. Here, we specify the coefficient matrices in (38) as $M = \mathrm{diag}(1,\ldots,50),\ R = 0.2 \times I_{50}$ and

$$K = \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}, \ F = H^\mathsf{T} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This system is depicted in Fig. 1, where we use the notation $q = [q_1,\ldots,q_{50}]^\mathsf{T}$. Furthermore, the Bode gain diagram of this system is plotted in Fig. 2 with the thin solid line. From this figure, we can see that the system has a number of resonance frequencies.

By applying Theorem 3, we approximate this system while preserving system dissipativity. More specifically, we aim to preserve $V$-dissipativity with respect to

$$Q = \begin{bmatrix} -1 & 0 \\ 0 & \gamma^2 \end{bmatrix}, \quad \gamma = \|G(s)\|_{\mathcal{H}_\infty} + 0.01 = 2.81.$$

This dissipativity preservation implies that the generalized singular perturbation approximant satisfies

$$\sup_{\omega\in\mathbb{R}} |\hat{G}_\sigma(j\omega)| = \|\hat{G}_\sigma(s)\|_{\mathcal{H}_\infty} < \gamma.$$

By varying the value $\epsilon$ in (37), which represents the threshold of neglected eigenvalues of $\Phi_\sigma$ in (31), we construct

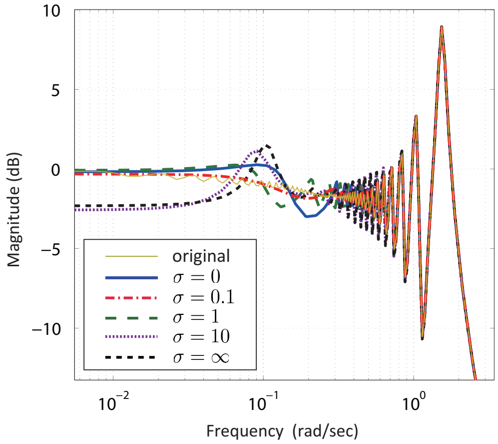


Fig. 2. Bode Gain Diagrams.

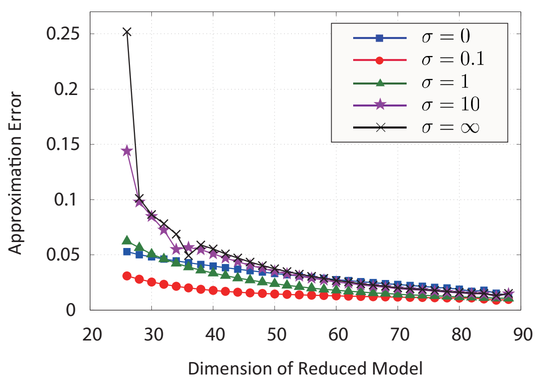


Fig. 3. Approximation Errors versus Dimension of Reduced Models.

generalized singular perturbation models. For several values of $\sigma \geq 0$, Fig. 3 shows the resultant approximation error versus the dimension of each approximate model. From this figure, we can see that the approximation error decreases as the dimension of each model increases. Since the dimension of approximate models is a decreasing function of $\epsilon$, this implies that the quality of approximate models can be appropriately captured by the neglected eigenvalues of $\Phi_\sigma$. Furthermore, we notice that the approximate model with $\sigma = 0.1$ gives the least approximation error among the values of $\sigma$ that we have tried.

Finally, the Bode gain diagram of each 30-dimensional approximate model is over-plotted in Fig. 2. From this figure, we can see that all approximate models appropriately capture the peak gain of the original system while the models with larger value of $\sigma$ (i.e., $\sigma = 10, \infty$) tend to cause larger approximation error in the low-frequency range. This trend can be recalled by the fact that the standard singular perturbation approximation (i.e., $\sigma = 0$) exactly preserves the zero frequency gain, while the projection-based model reduction (i.e., $\sigma = \infty$) preserves the infinite frequency gain. It is found that the approximate model with $\sigma = 0.1$ most appropriately captures overall frequency properties of the original system.

## IV. Conclusion

In this paper, based on a notion of generalized singular perturbation approximation, we have developed a model reduction method that preserves system dissipativity. It has been found that the generalized singular perturbation approximation can deal with not only the standard singular perturbation approximation but also the projection-based model reduction as a special case. In this sense, this model reduction provides a unified framework for major model reduction methods. Finally, the efficiency of the model reduction has been shown through an example of second-order systems.

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