Dissipativity-Preserving Model Reduction Based on Generalized Singular Perturbation

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Introduction

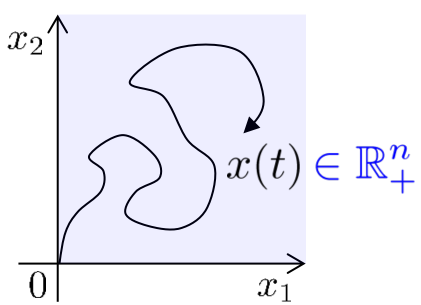
- Classical model reduction problem
  - Given $G(s)$, find $\hat{G}(s)$ such that $\|G-\hat{G}\| \le \epsilon$

Balanced truncation

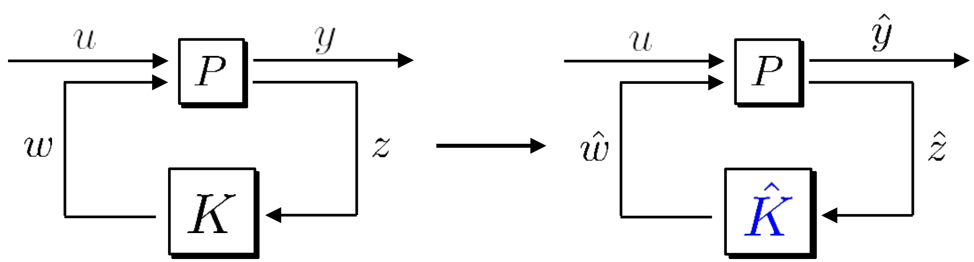
Hankel norm approximation

- Structure-preserving model reduction problem
  - Given $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{S}_n$, find $\hat{G}(s) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] \in \mathcal{S}_{\hat{n}}$ s.t. $\|G-\hat{G}\| \le \epsilon$
  - $\mathcal{S}_n$: a class of realizations that possess *particular structure(s)*

Positivity preservation



Controller reduction



Structure preservation is crucial for *practical* approximations

Model Reduction Based on Singular Perturbation

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$\begin{bmatrix} P \\ \overline{P} \end{bmatrix} x = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

unitary transform

$$\tilde{\Sigma} : \begin{cases} \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} PAP^{\mathsf{T}} & PA\overline{P}^{\mathsf{T}} \\ \overline{P}AP^{\mathsf{T}} & \overline{P}A\overline{P}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} PB \\ \overline{P}B \end{bmatrix} u \\ y = \begin{bmatrix} CP^{\mathsf{T}} & C\overline{P}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + Du \end{cases}$$

Impose $\dot{\eta} \equiv 0$ on $\dot{\eta} = \overline{P}AP^{\mathsf{T}}\xi + \overline{P}A\overline{P}^{\mathsf{T}}\eta + \overline{P}Bu$

Singular perturbation approximation

✓ $\Pi := -\overline{P}^{\mathsf{T}}(\overline{P}A\overline{P}^{\mathsf{T}})^{-1}\overline{P}$

$$\hat{\Sigma} : \begin{cases} \dot{\hat{\xi}} = \hat{A}\hat{\xi} + \hat{B}u \\ \hat{y} = \hat{C}\hat{\xi} + \hat{D}u \end{cases}$$

where

$$\begin{cases} \hat{A} := PAP^{\mathsf{T}} + PA\Pi AP^{\mathsf{T}} \\ \hat{B} := PB + PA\Pi B \\ \hat{C} := CP^{\mathsf{T}} + C\Pi AP^{\mathsf{T}} \\ \hat{D} := D + C\Pi B \end{cases}$$

Model Reduction Based on Generalized Singular Perturbation

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

$$\begin{bmatrix} P \\ \overline{P} \end{bmatrix} x = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

unitary transform

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Impose $\dot{\eta} \equiv \sigma\eta$ on $\dot{\eta} = \overline{P}AP^{\mathsf{T}}\xi + \overline{P}A\overline{P}^{\mathsf{T}}\eta + \overline{P}Bu$

Generalized singular perturbation approximation

Introducing parameter σ

✓ $\Pi := \overline{P}^{\mathsf{T}}(\sigma I_{n-\hat{n}} - \overline{P}A\overline{P}^{\mathsf{T}})^{-1}\overline{P}$

$$\hat{\Sigma}_{\sigma} : \begin{cases} \dot{\hat{\xi}} = \hat{A}\hat{\xi} + \hat{B}u \\ \hat{y} = \hat{C}\hat{\xi} + \hat{D}u \end{cases} \quad \text{where} \quad \begin{cases} \hat{A} := PAP^{\mathsf{T}} + PA\Pi AP^{\mathsf{T}} \\ \hat{B} := PB + PA\Pi B \\ \hat{C} := CP^{\mathsf{T}} + C\Pi AP^{\mathsf{T}} \\ \hat{D} := D + C\Pi B \end{cases}$$

$\hat{\Sigma}_{\sigma}$ coincides with
- singular perturbation approximant if $\sigma = 0$
- projection-based $(PAP^{\mathsf{T}}, PB, CP^{\mathsf{T}}, D)$ if $|\sigma| \to \infty$

Problem Formulation

$\mathcal{S}_n$: a class of n-dimensional systems that possess ***structure(s)***

e.g., dissipativity, network structure among subsystems

in the following

[Problem]

Consider a system $\Sigma \in \mathcal{S}_n$. Given $\delta \geq 0$, find a generalized singular perturbation model $\hat{\Sigma}_\sigma \in \mathcal{S}_{\hat{n}}$ associated with $P \in \mathcal{P}^{\hat{n}\times n}$ such that

$$\|G(s) - \hat{G}_\sigma(s;P)\|_{\mathcal{H}_2} \leq \delta.$$

- ✓ set of projections $\mathcal{P}^{\hat{n}\times n} := \{P \in \mathbb{R}^{\hat{n}\times n} : PP^\mathsf{T} = I_{\hat{n}},\ \hat{n} \leq n\}$
- ✓ transfer matrices of Σ and $\hat{\Sigma}_\sigma$:

$$G(s) := C(sI_n - A)^{-1}B + D$$

$$\hat{G}_\sigma(s;P) := \hat{C}(sI_{\hat{n}} - \hat{A})^{-1}\hat{B} + \hat{D} \qquad \begin{cases} \hat{A} = PAP^\mathsf{T} + PA\Pi AP^\mathsf{T} \\ \hat{B} = PB + PA\Pi B \\ \hat{C} = CP^\mathsf{T} + C\Pi AP^\mathsf{T} \\ \hat{D} = D + C\Pi B \end{cases}$$

System Dissipativity

[Definition]

A system Σ is said to be <u>V-*dissipative with respect to* $Q = Q^\mathsf{T}$</u> if

there exists $V \succ \mathcal{O}_n$ such that $\mathcal{F}_Q(A,B,C,D;V) \prec \mathcal{O}_{n+m_u}$ for

$$\mathcal{F}_Q(A,B,C,D;V) := \begin{bmatrix} A^\mathsf{T}V + VA & VB \\ B^\mathsf{T}V & 0 \end{bmatrix} - \begin{bmatrix} C^\mathsf{T} & 0 \\ D^\mathsf{T} & I_{m_u} \end{bmatrix} Q \begin{bmatrix} C & D \\ 0 & I_{m_u} \end{bmatrix}.$$

✓ equivalent to $\frac{d}{dt}(x^\mathsf{T}Vx) < \begin{bmatrix} y^\mathsf{T} & u^\mathsf{T} \end{bmatrix} Q \begin{bmatrix} y \\ u \end{bmatrix}$ along trajectory of Σ

[Lemma]

Any V-dissipative Σ has a realization that is I_n-dissipative w.r.t. Q .

✓ Cholesky factorization $V = V_{\frac{1}{2}}^\mathsf{T} V_{\frac{1}{2}}$: $x^\mathsf{T}Vx = (V_{\frac{1}{2}}x)^\mathsf{T} I_n (V_{\frac{1}{2}}x)$

Dissipativity Preservation

[Theorem]

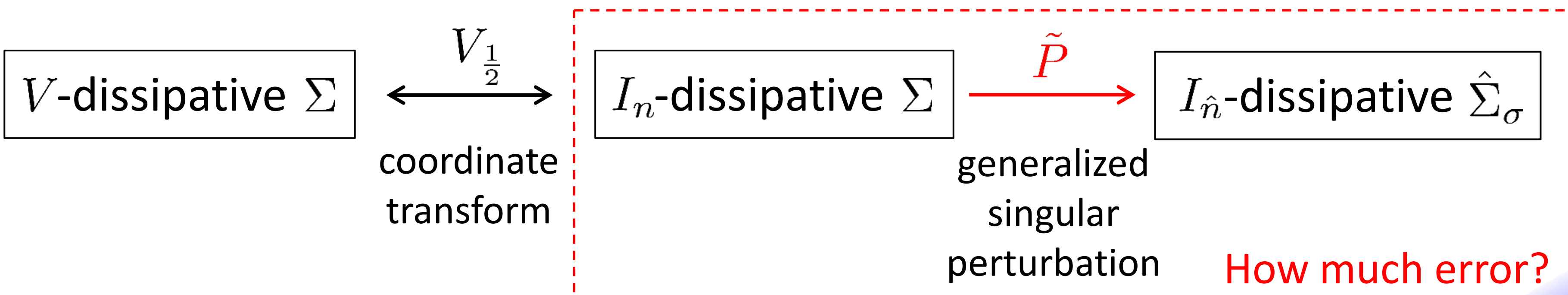
Let Σ be given, and suppose that it is I_n-dissipative w.r.t. Q.

If $\sigma \geq 0$ and $P \in \mathcal{P}^{\hat{n} \times n}$ satisfies $\mathrm{im}(C^{\mathsf{T}}) \subseteq \mathrm{im}(P^{\mathsf{T}})$,

then $\hat{\Sigma}_\sigma$ is $I_{\hat{n}}$-dissipative w.r.t. Q.

✓ If $\mathrm{im}(C^{\mathsf{T}}) \subseteq \mathrm{im}(P^{\mathsf{T}})$, $\hat{\Sigma}_\sigma$ admits *projection-like* formula:

$$(\hat{A}, \hat{B}, \hat{C}, \hat{D}) = \left(\tilde{P} A \tilde{P}^{\mathsf{T}} - \sigma P A \Pi (P A \Pi)^{\mathsf{T}}, \tilde{P} B, C \tilde{P}^{\mathsf{T}}, D\right) \text{ where } \tilde{P} := P + P A \Pi.$$

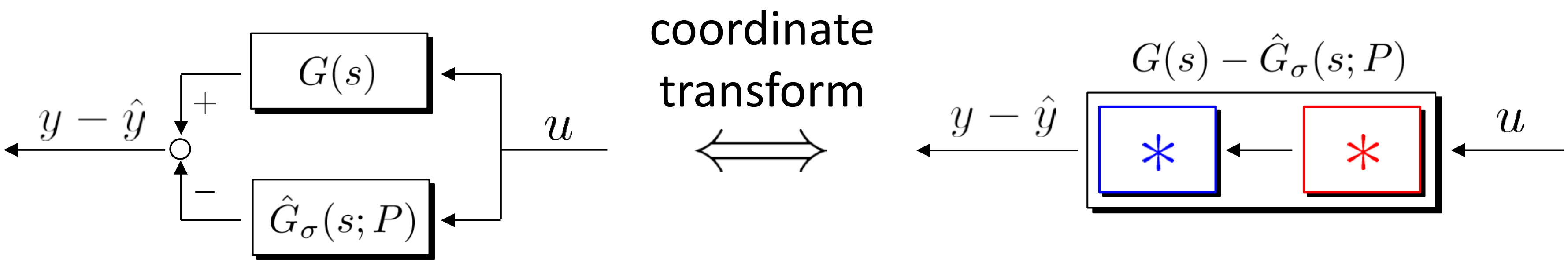


Error Analysis

[Theorem] The error system can be factorized as

$$G(s) - \hat{G}_\sigma(s;P) = \begin{cases} \hat{\Xi}_\sigma(s;P)\overline{P}^{\mathsf{T}}\overline{P}X_\sigma(s) \\ (\hat{\Xi}_\sigma(s;P)A + C)\overline{P}^{\mathsf{T}}\sigma^{-1}\overline{P}X_\sigma(s), & \text{if } \sigma \neq 0 \end{cases}$$

where
$$\begin{cases} \hat{\Xi}_\sigma(s;P) := \hat{C}(sI_{\hat{n}} - \hat{A})^{-1}(P + PA\Pi) + C\Pi \\ X_\sigma(s) := (\sigma I_n - A)(sI_n - A)^{-1}B - B. \end{cases}$$



✓ Expect: error $y - \hat{y}$ will be small if $P \in \mathcal{P}^{n \times \hat{n}}$ is chosen so that

$\|\overline{P}X_\sigma(s)\|$ is sufficiently small (while $\| * \|$ is bounded)

Error Analysis

[Theorem] The error system can be factorized as

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where $\begin{cases} \hat{\Xi}_\sigma(s;P) := \hat{C}(sI_{\hat{n}} - \hat{A})^{-1}(P + PA\Pi) + C\Pi \\ X_\sigma(s) := (\sigma I_n - A)(sI_n - A)^{-1}B - B. \end{cases}$

[Theorem] ✓ $\|*\|_{\mathcal{H}_\infty} < \gamma$ & $\|*\|_{\mathcal{H}_2} \leq \epsilon$

Let $\gamma > 0$ such that $\mathcal{O}_n \succ \begin{cases} A + A^\mathsf{T} + \gamma^{-1}(I_n + C^\mathsf{T}C) \\ A + A^\mathsf{T} + \gamma^{-1}(AA^\mathsf{T} + C^\mathsf{T}C), & \text{if } \sigma \neq 0. \end{cases}$

Define $\Phi_\sigma := (\sigma I_n - A)W(\sigma I_n - A)^\mathsf{T}$ for W such that $AW + WA^\mathsf{T} + BB^\mathsf{T} = 0.$

If $\mathrm{im}([B, C^\mathsf{T}]) \subseteq \mathrm{im}(P^\mathsf{T})$ and $\epsilon \geq \begin{cases} \sqrt{\mathrm{tr}(\Phi_\sigma) - \mathrm{tr}(P\Phi_\sigma P^\mathsf{T})} \\ |\sigma|^{-1}\sqrt{\mathrm{tr}(\Phi_\sigma) - \mathrm{tr}(P\Phi_\sigma P^\mathsf{T})}, & \text{if } \sigma \neq 0, \end{cases}$

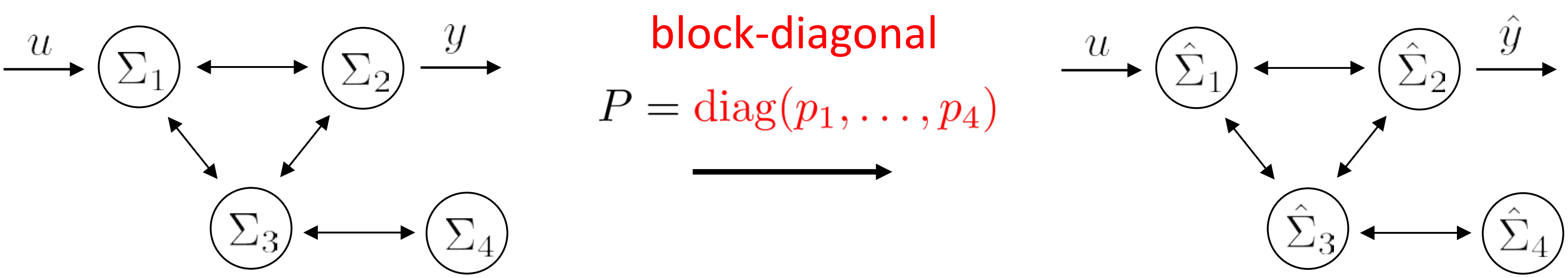
then $$\|G(s) - \hat{G}_\sigma(s;P)\|_{\mathcal{H}_2} \leq \gamma\epsilon.$$

Eigenvalues of Φ_σ is related to approximation error

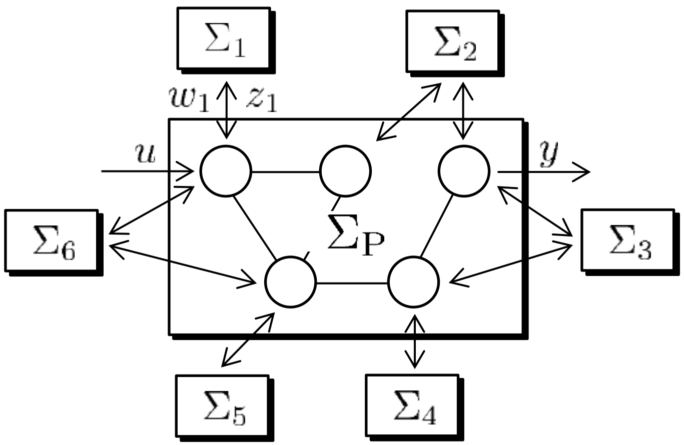
Remarks

- Eigenvalue decomposition of $\Phi_\sigma = (\sigma I_n - A) W (\sigma I_n - A)^{\mathsf{T}}$
  - leads to $P \in \mathcal{P}^{n \times \hat{n}}$ that achieves $\|G(s) - \hat{G}_\sigma(s; P)\|_{\mathcal{H}_2} \leq \gamma \epsilon$ for given ϵ
  - ϵ : design criterion to regulate approximating quality

- Generalization to <u>network structure preservation</u>
  - allows application to controller reduction problem

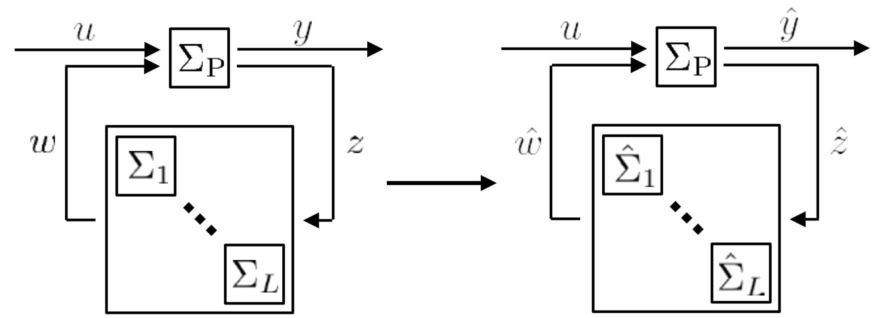


Application to Structured Passive Controller Reduction



Σ_P : Passive plant

Σ_l : Passive controller



Original Closed-Loop $(\Sigma_\mathrm{P}, \{\Sigma_l\}_{l\in\mathbb{L}})$

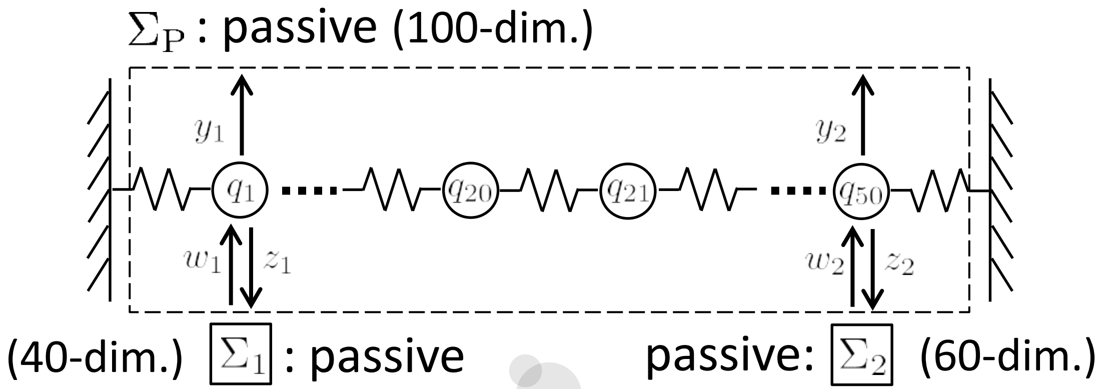
Approximate Model $(\Sigma_\mathrm{P}, \{\hat{\Sigma}_l\}_{l\in\mathbb{L}})$

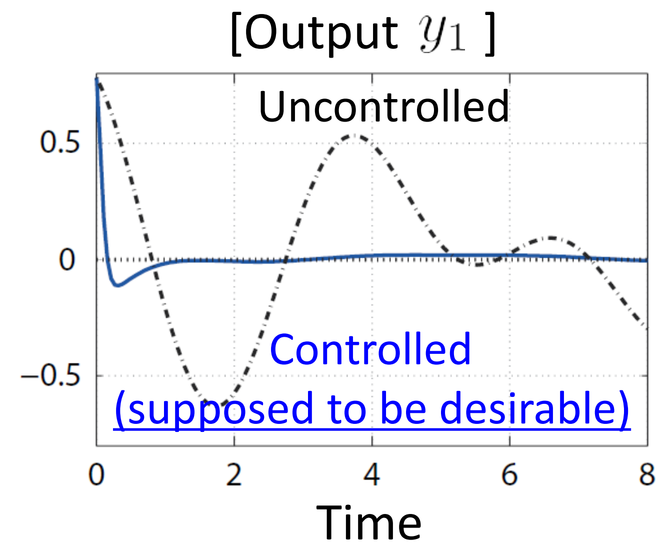
By virtue of dissipativity & network structure preservation:

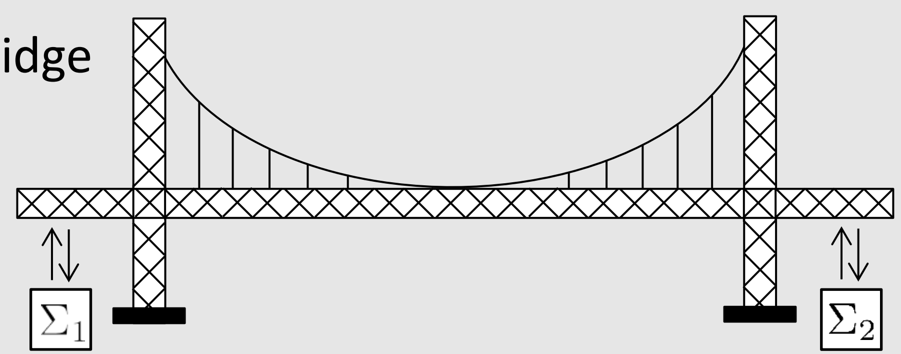
Find an approximate model $(\Sigma_\mathrm{P}, \{\hat{\Sigma}_l\}_{l\in\mathbb{L}})$ such that each $\hat{\Sigma}_l$ remains passive and $y-\hat{y}$ is small enough in $\mathcal{H}_2$-sense.

Numerical Example

Vibration suppression control:





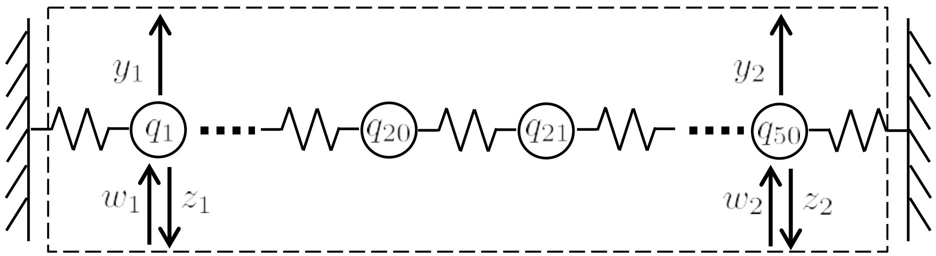


Limited sensor & actuator allocations

Numerical Example

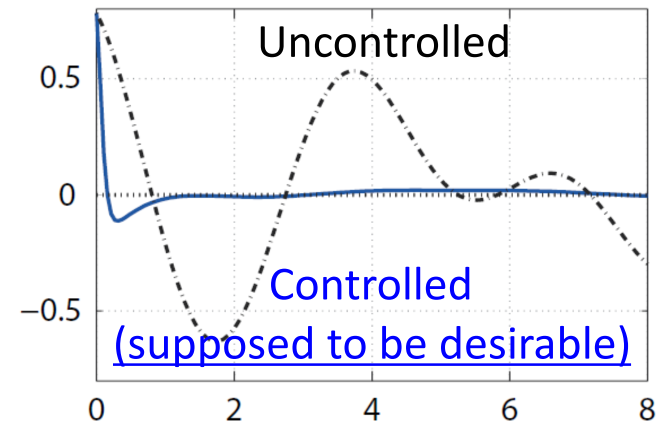
Vibration suppression control:

Σ_{P} : passive (100-dim.)

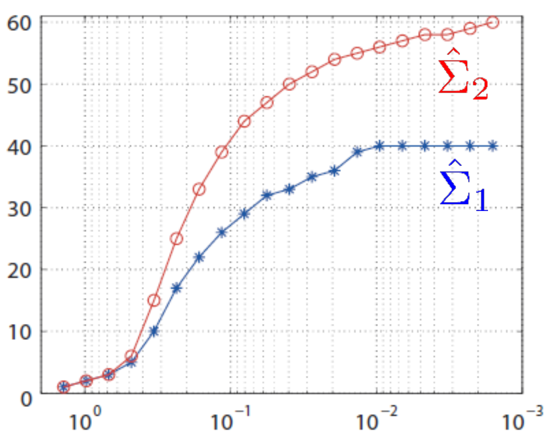


(40-dim.) Σ_1 : passive   passive: Σ_2 (60-dim.)

[Output y_1]

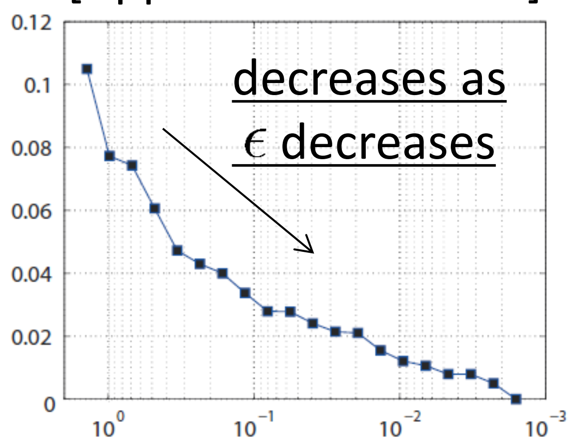


[Dim. of $\hat{\Sigma}_1$ & $\hat{\Sigma}_2$]



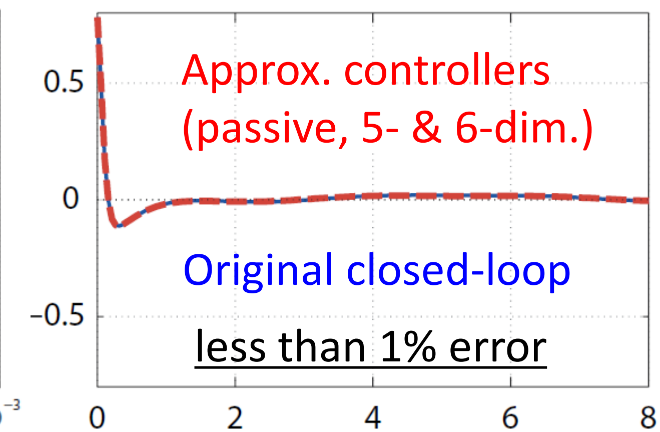
ϵ (design criterion)

[Approximation error]



ϵ (design criterion)

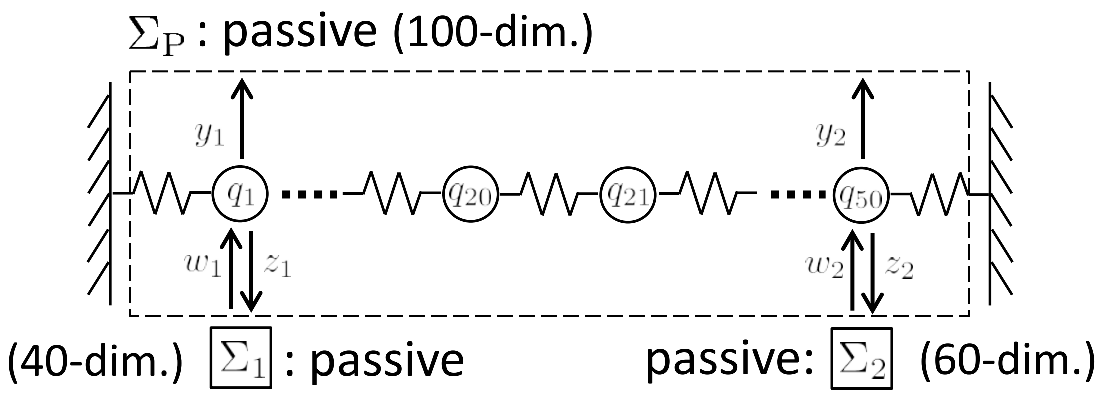
[Output $\hat{y}_1$ of approx. model]



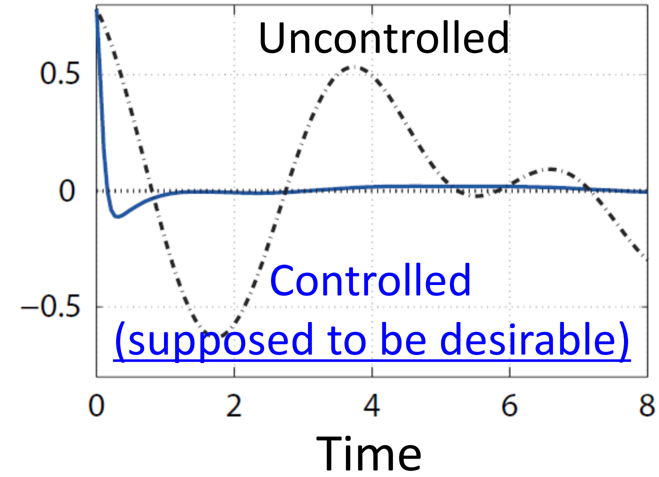
Time

Numerical Example

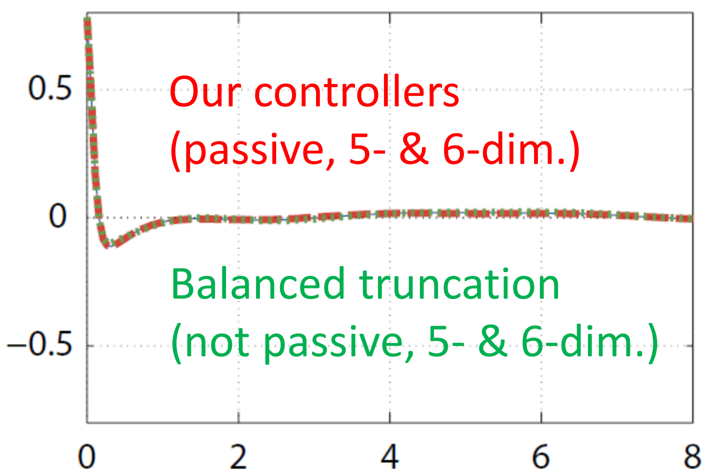
Vibration suppression control:



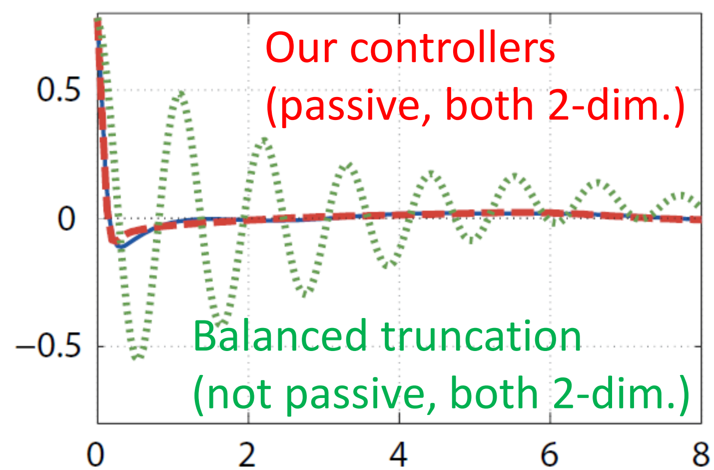
[Output y_1]



[Output $\hat{y}_1$ of approx. models]



[Output $\hat{y}_1$ of approx. models]



Concluding Remarks

- Generalized singular perturbation approximation includes:
  - standard singular perturbation
  - projection-based model reduction
- Structure-preserving model reduction
  - dissipativity, network structure among subsystems
  - a priori $\mathcal{H}_2$-error bound
- Application to distributed passive controller reduction
  - vibration suppression for interconnected second-order systems
  - preservation of passivity and closed-loop performance

Thank you for your attention!