Extraction of 1-Dimensional Reaction-Diffusion Structure in SISO Linear Dynamical Networks

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***Abstract*— In this paper, we propose a model order reduction method for SISO linear dynamical networks, where a large number of subsystems are interconnected according to a network. In this method, the structure of spatially one-dimensional reaction-diffusion that a SISO linear dynamical network has is extracted by way of Householder transformation ordering the state variables according to the distance from the source (i.e., an input) of the diffusion. Based on this structure, a model order reduction method with the diffusion structure of the system preserved is presented, which can be applied for large-scale systems. In addition, an easily-computable error bound via the proposed model reduction is derived.**

I. Introduction

Dynamical systems on large-scale/complex networks, whose behaviors are determined by the interaction of a large number of subsystems, have been intensively studied. Examples of such dynamical networks include World Wide Web, gene regulatory networks, spread of infection. In these systems, the signal transmissions on a network can be interpreted as a kind of diffusion phenomena on the network. Thus, various works on diffusion phenomena of dynamical systems on large-scale/complex networks have been performed (see e.g., [1], [2], [3]).

In terms of such diffusion phenomena, this paper addresses the model reduction issue of SISO linear systems on large-scale networks expressed by undirected graphs, whose nodes and edges denote subsystems and their interactions, respectively. As shown in Fig 1, signal transmissions from a node with an external input, i.e., a node of diffusion source, on a network are regarded as a kind of spatially *one-dimensional* reaction-diffusion. Therefore, if such a one-dimensional diffusion structure is extracted from the system in question, it can be expected that the complexity of the system is reduced by ignoring the nodes (i.e., state variables) on which the effect of the external input is relatively small, while keeping the diffusion structure. To this end, we use Householder transformation, which is effective for tri-diagonalization of large-scale (symmetric) matrices. Thus, the proposed model reduction method is effective for large-scale SISO linear systems on undirected graphs.

Many kinds of model reduction methods of linear and nonlinear systems have been developed [4], [5], [6]. However, the singular value decomposition (SVD) method such as balanced model reduction cannot in general preserve the above diffusion structure of the original system, although the stability/passivity property is preserved [4], [5]. In addition, this method often require computationally expensive operations such as balanced realizations. Therefore, these methods are not in general suitable for large-scale systems. On the other hand, Krylov method is also well-known as a model reduction method for large-scale systems. However, in this method, the stability property is not in general preserved, and a priori computable error bound caused via model reduction has not been derived even in linear systems [4]. Although the reduced model of the proposed method has the mathematical structure similar to that of Krylov method, the proposed method, by focusing on the class of reaction-diffusion systems, has the stability preservation and provides an explicit error bound caused via model reduction, which can be easily obtained by iterative matrix computations. Furthermore, the spatially one-dimensional reaction-diffusion structure is determined for this class of systems.

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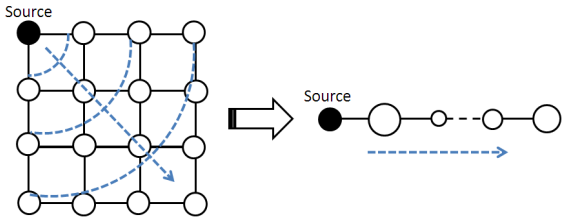

Fig. 1. Regarding diffusion on square graph as one-dimensional diffusion

This paper is organized as follows. In section II, we describe a system to be studied here and present a coordinate transformation to extract a one-dimensional reaction-diffusion structure from the system. In section III, we analyze the properties of the system with the reaction-diffusion structure and propose a model order reduction method. Section IV shows the validity of the method by numerical examples including reaction-diffusion systems on a square lattice and a complex dynamical network. Finally section V concludes this paper.

NOTATION: Let v be a (column or row) vector and $M = \{m_{ij}\}$ a matrix.

$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of non-negative real numbers
I_n	the unit matrix of the size $n \times n$
$v\,(p:q)$	the vector formed by the p- to q-th entries of v
$M\,(p:q, r:s)$	the matrix formed by the p- to q-th rows and the r- to s-th columns of M
$\mathrm{abs}\,(M)$	the matrix defined by $\{\lvert m_{ij}\rvert\}$

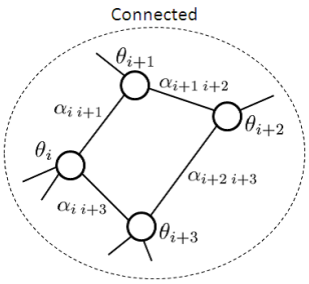

Fig. 2. Graph structure of target system

II. Extraction of Reaction-Diffusion Structure

A. System Description

In this paper, we deal with SISO systems that evolve on large-scale complex networks. The connection structure of the networks is described by an undirected graph composed of n nodes. The state of i-th node is denoted by θ_i and the scalar input and output of the system are denoted by u and y. Then, the state equation is given by

$$\left\{\begin{array}{l} \dot{\theta}_i = -\gamma_i\theta_i + \displaystyle\sum_{j=1,j\neq i}^{n} \alpha_{ij}\left(\theta_j-\theta_i\right)+\beta_i u \\ y = \displaystyle\sum_{i=1}^{n}\eta_i\theta_i \end{array}\right. \tag{1}$$

$\gamma_i \geq 0$ with at least one strict inequality

$\alpha_{ij} = \alpha_{ji} \geq 0,\ i \neq j$

$\forall i,j,\ \exists p_1,\cdots,p_k \text{ s.t. } \alpha_{ip_1}\alpha_{p_1p_2}\cdots\alpha_{p_{k-1}p_k}\alpha_{p_kj} > 0$

where α_{ij} and γ_i denote the intensity of the diffusion between i-th and j-th nodes, and the non-negative (self-decomposition) reaction coefficient of i-th node; see Fig. 2. The last condition represents the connectivity of the graph.

By taking a state vector $\theta := \left[\begin{array}{ccc}\theta_n & \cdots & \theta_1\end{array}\right]^{\mathsf{T}}$, we obtain the system matrices of the state-space representation (A,B,C) given by

$$\begin{array}{l} A := \{A_{ij}\} \\ \left\{\begin{array}{ll} A_{ij} = \alpha_{ij} & (i\neq j) \\ A_{ii} = -\gamma_i - \displaystyle\sum_{j=1,j\neq i}^{n}\alpha_{ij} & (i=j)\,, \end{array}\right. \\ B = \left[\begin{array}{ccc}\beta_n & \cdots & \beta_1\end{array}\right]^{\mathsf{T}}, \\ C = \left[\begin{array}{ccc}\eta_n & \cdots & \eta_1\end{array}\right], \end{array} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$. The symmetric matrix A is negative definite, namely the system is stable.

B. Reaction-diffusion structure extraction

In this section, we introduce a class of state-space representations that play a crucial role in this paper. These representations are closely related to a reaction-diffusion property.

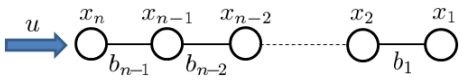

Fig. 3. Illustration of **RD**-representation

Definition 1: System matrices $(\mathcal{A},\mathcal{B},\mathcal{C})$ are said to have a *reaction-diffusion structure with boundary input* if

$$\begin{array}{rcl} \mathcal{A} & = & \left[\begin{array}{ccccc} a_n & b_{n-1} & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & b_1 \\ & & & b_1 & a_1 \end{array}\right] \\ \mathcal{B} & = & \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array}\right]^{\mathsf{T}} \end{array} \tag{3}$$

for some $a_i < 0$ and $b_i \geq 0$ for all $i = 1,\cdots,n$.

Hereafter, a representation by system matrices that have the reaction-diffusion structure with boundary input is denoted as a **RD**-representation and its state vector and $\mathcal{C}$ are denoted as $x := \left[\begin{array}{ccc} x_n & \cdots & x_1 \end{array}\right]^{\mathsf{T}}$ and $\mathcal{C} := \left[\begin{array}{ccc} c_n & \cdots & c_1 \end{array}\right]$. In particular, if $b_i > 0$ for all i, we say **RDp**-representation. This structure represents autonomous systems diffusively coupled in a series cascade manner equipped with the boundary input; see Fig. 3. Actually, **RD**-representations pop up when we apply the finite difference method to spatially one-dimensional partial differential reaction-diffusion systems with a boundary input [7], [8], [9].

Theorem 1: The **RD**-representation of the system (1) is given by

$$\mathcal{A} = H_+^{\mathsf{T}}V^{\mathsf{T}}AVH_+,\quad \mathcal{C} = \|B\|\,CVH_+, \tag{4}$$

where V is an orthogonal matrix with $B/\|B\|$ in the first column and H_+ is a Householder matrix that makes $\mathcal{A}$ in (4) tri-diagonal with the all off-diagonal entries being non-negative. Moreover, $\bar{i} := \min_i\{b_{n-i}=0\}$, $\mathcal{A}\left(1:\bar{i},1:\bar{i}\right)$ and $\mathcal{C}\left(1:\bar{i}\right)$ are unique in **RD**-representations obtained by any orthogonal transformations.

Proof: [Proof of **RD**-representation] We first show that (4) gives a **RD**-representation of the system (1). See [10] for details of the Householder transformation. Note that the transformation VH_+ is orthogonal. By direct computation, we obtain $H_+^{\mathsf{T}}V^{\mathsf{T}}B = \left[\begin{array}{cccc}\|B\| & 0 & \cdots & 0\end{array}\right]^{\mathsf{T}}$. Furthermore, the transformation from A to $\mathcal{A}$ preserves the symmetry and negative definiteness. Hence, the desired result follows.

[Proof of uniqueness] We move to prove the uniqueness. From the structure of matrices in **RD**-representation, $\bar{i}$ is equal to the dimension of the controllable subspace, and consequently is independent of representations.

Suppose that an orthogonal matrix P makes $(\mathcal{A},\mathcal{B},\mathcal{C})$ and $\left(P^{\mathsf{T}}\mathcal{A}P, P^{\mathsf{T}}\mathcal{B}, \mathcal{C}P\right)$ be **RD**-representation. It suffices to show

$$P\left(1:\bar{i},1:\bar{i}\right) = I_{\bar{i}},\ P\left(\bar{i}+1:n,1:\bar{i}\right) = 0,\ P\left(1:\bar{i},\bar{i}+1:n\right) = 0.$$

This can be shown by direct calculations of $P^{\mathsf{T}}\mathcal{A}P$ and $P^{\mathsf{T}}\mathcal{B}$ (the detail is omitted due to space limitations). ■

As in the proof, $x\left(\bar{i}+1:n\right)$ is in the uncontrollable subspace that is irrelevant in the sense of input/output mapping. Therefore, Theorem 1 means that by computing (4) and eliminating $x\left(\bar{i}+1:n\right)$, we can extract the unique **RDp**-representation obtained by the orthogonal transformations.

It should be emphasized that the construction of the Householder matrices does not require computationally expensive operations, e.g., singular decompositions. Moreover, constructing methods of the tri-diagonalizing matrix for large matrices have been actively studied in the computer science since Householder transformations are often applied when finding the eigenvalues [10]. In this sense, this transformation can be implemented even for large-scale systems.

III. Analysis of Systems given by Reaction-Diffusion representation with boundary input

A. Retroactive Series Cascade Structure

In this section, we analyze some properties of the **RD**-representation and propose a method of model order reduction. Since the uncontrollable modes are irrelevant in the sense of input/output mapping, we hereafter consider systems with **RDp**-representation.

Theorem 2: Consider a SISO system with an input u, a state vector $x = \left[\begin{array}{ccc} x_n & \cdots & x_1 \end{array}\right]^{\mathsf{T}}$ and a **RDp**-representation $(\mathcal{A},\mathcal{B},\mathcal{C})$ in (3) with a stable $\mathcal{A}$. Then, the transfer function from u to x_i is given by

$$G_i(s) = \prod_{k=i}^{n} S_k(s) \tag{5}$$

where

$$\left\{\begin{array}{l} S_{i+1}(s) = \dfrac{b_{i+1}}{s - a_{i+1} - b_i S_i(s)} \\ S_1(s) = \dfrac{b_1}{s-a_1}. \end{array}\right. \tag{6}$$

In particular,

1) G_i is stable and has the relative degree $n-i+1$.
2) S_i is stable and $|S_i(j\omega)|$ is monotonically decreasing with respect to $\omega \in \mathbb{R}_+$.
3) For an integer j such that $1 \le j \le n-1$, let us define $S_i(s;b_j)$ and $G_i(s;b_j)$ by regarding b_j as a variable. Then, for $i > j$

$$\frac{\partial S_i(j\omega;b_j)}{\partial b_j} = \frac{2}{b_i} S_j(j\omega;b_j) \prod_{k=j+1}^{i} S_k^2(j\omega;b_j) \tag{7}$$

holds.

Proof: For $i \le n-1$, from the state equation with respect to each state variable, we have $x_i = S_i(s)\, x_{i+1}$ and for $i = n$ we have $x_n = S_n(s)\, u$, where $b_n = 1$. Therefore we obtain (5).

[Proof of 1)] Let us denote the numerator and denominator polynomial of S_i by N_i and D_i, respectively. Then, we obtain

$$\left\{\begin{array}{l} N_{i+1}(s) = b_{i+1} D_i(s) \\ D_{i+1}(s) = (s-a_{i+1}) D_i(s) - b_i N_i(s), \end{array}\right. \tag{8}$$

and consequently the polynomial sequence

$$\left\{\begin{array}{l} D_{i+2}(s) = (s-a_{i+2}) D_{i+1}(s) - b_{i+1}^2 D_i(s) \\ D_0(s) = 1 \\ D_1(s) = s - a_1. \end{array}\right. \tag{9}$$

Therefore, we obtain

$$G_i(s) = \prod_{k=i}^{n} S_k(s) = \left(\prod_{k=i}^{n} b_k\right) \frac{D_{i-1}(s)}{D_n(s)}. \tag{10}$$

Since D_i is i-th order polynomial, the relative degree of G_i is $n-i+1$.

[Proof of 2)] The polynomial sequence $D_n, \cdots, D_0$ defined by (9) makes a Strum sequence (see [10] for details). Therefore, from the zero of D_{i-1} exists between the two zeros of D_i, which are next to each other, the result follows.

[Proof of 3)] Equation (6) yields

$$\begin{aligned} \frac{\partial S_{j+1}(j\omega;b_j)}{\partial b_j} &= \frac{\partial}{\partial b_j}\left\{\frac{b_{j+1}}{j\omega - a_{j+1} - b_j S_j(j\omega;b_j)}\right\} \\ &= \frac{2}{b_{j+1}} S_{j+1}^2(j\omega;b_j)\, S_j(j\omega;b_j) \end{aligned} \tag{11}$$

and for $i \ge j+1$

$$\begin{aligned} \frac{\partial S_{i+1}(j\omega;b_j)}{\partial b_j} &= \frac{b_{i+1} b_i}{\{j\omega - a_{i+1} - b_i S_i(j\omega;b_j)\}^2} \frac{\partial S_i}{\partial b_j} \\ &= \frac{b_i}{b_{i+1}} S_{i+1}^2(j\omega;b_j) \frac{\partial S_i(j\omega;b_j)}{\partial b_j}. \end{aligned} \tag{12}$$

By applying these equalities repeatedly, we obtain the desired equality. ■

From Theorem 2-1), we can see that G_i has the low-pass property with $(n-i+1)$-th order. More specifically, **RDp**-representations represent a series cascade consisting of low-pass filters. Note that the S_i given in (6) are determined retroactively in that the variation of b_i interferes S_j for all $j > i$ as shown in Theorem 2-3).

B. Application to Model Order Reduction

In this section, we propose a model reduction method based on **RDp**-representation. The idea is simple: eliminate $x(k+1:n)$ in order to obtain k-dimensional model.

Theorem 3: For all i and $1 \le j \le n-1$

$$\left\|G_i(s;b_j) - G_i(s;0)\right\|_\infty = G_i(0;b_j) - G_i(0;0) \ge 0. \tag{13}$$

Proof: Trivially $G(s;0) = 0$ for all $i \le j$ since $b_j = 0$ implies $S_j = 0$. Furthermore, $|G_i(j\omega)|$ is monotonically decreasing with respect to ω from Theorem 2. Therefore, the result holds for all $i \le j$. In what follows, we assume $i > j$.

Suppose the following inequalities hold:

$$\left|\frac{\partial G_i(j\omega;b_j)}{\partial b_j}\right| \le \left|\frac{\partial G_i(0;b_j)}{\partial b_j}\right|, \tag{14}$$

$$\frac{\partial G_i(0;b_j)}{\partial b_j} \ge 0. \tag{15}$$

Then, we obtain

$$
\begin{aligned}
&|G_i(j\omega;b_j) - G_i(j\omega;0)| = \left| \int_0^{b_j} \frac{\partial G_i(j\omega;b)}{\partial b} db \right| \\
&\le \int_0^{b_j} \left| \frac{\partial G_i(0;b)}{\partial b} \right| db = \int_0^{b_j} \frac{\partial G_i(0;b)}{\partial b} db \\
&= G_i(0;b_j) - G_i(0;0). \qquad (16)
\end{aligned}
$$

This readily implies (13).

What remains to be shown is (14) and (15). Note that $S_i(0;b_j) > 0$. By differentiating $G_i(j\omega;b_j) = \prod_{k=i}^{n} S_k(j\omega;b_j)$ and using Theorem 2-2) and 2-3), we have

$$
\begin{aligned}
&\frac{\partial G_i(j\omega;b_j)}{\partial b_j} = \sum_{l=i}^{n} \left\{ \frac{\partial S_l}{\partial b_j} \prod_{m=i,m\neq l}^{n} S_m(j\omega;b_j) \right\} \\
&= \frac{2}{b_i} S_j(j\omega;b_j) \sum_{l=i}^{n} \left\{ \prod_{k=j+1}^{l} S_k^2(j\omega;b_j) \prod_{m=i,m\neq l}^{n} S_m(j\omega;b_j) \right\},
\end{aligned}
$$

which directly implies (15). Moreover

$$
\begin{aligned}
&\left| \frac{\partial G_i(j\omega;b_j)}{\partial b_j} \right| \\
&\le \left| \frac{2}{b_i} S_j(j\omega;b_j) \sum_{l=i}^{n} \left\{ \prod_{k=j+1}^{l} S_k^2(j\omega;b_j) \prod_{m=i,m\neq l}^{n} S_m(j\omega;b_j) \right\} \right| \\
&\le \left| \frac{2}{b_i} S_j(0;b_j) \right| \sum_{l=i}^{n} \left| \prod_{k=j+1}^{l} S_k^2(0;b_j) \prod_{m=i,m\neq l}^{n} S_m(0;b_j) \right| \\
&= \left| \frac{2}{b_i} S_j(0;b_j) \sum_{l=i}^{n} \left\{ \prod_{k=j+1}^{l} S_k^2(0;b_j) \prod_{m=i,m\neq l}^{n} S_m(0;b_j) \right\} \right| \\
&= \left| \frac{\partial G_i(0;b_j)}{\partial b_j} \right|. \qquad (17)
\end{aligned}
$$

Thus, (14) follows. ■

This Theorem enables us to estimate the upper bound of the error caused by eliminating $x(j+1:n)$:

Theorem 4: Let $(\mathcal{A},\mathcal{B},\mathcal{C})$ be system matrices of **RDp**-representation. Denote $G(s) = \mathcal{C}(sI_n - \mathcal{A})^{-1}\mathcal{B}$ and $\hat{G}_k(s) = \mathcal{C}_k(sI_k - \mathcal{A}_k)^{-1}\mathcal{B}_k$, where $\mathcal{A}_k := \mathcal{A}(1:k,1:k)$, $\mathcal{B}_k := \mathcal{B}(1:k)$ and $\mathcal{C}_k := \mathcal{C}(1:k)$. If $\mathcal{A}$ is stable, then $\mathcal{A}_k$ is stable and

$$
\left\| G(s) - \hat{G}_k(s) \right\|_\infty \le \left| \mathrm{abs}(\mathcal{C}) \mathcal{A}^{-1}\mathcal{B} - \mathrm{abs}(\mathcal{C}_k) \mathcal{A}_k^{-1}\mathcal{B}_k \right| \qquad (18)
$$

holds. Moreover, if the entries of $\mathcal{C}$ have the same sign, (18) holds with equality.

Proof: The proof of stability of $\mathcal{A}_k$ is trivial thanks to the negative definiteness of $\mathcal{A}$. Noting $G_i(j\omega;b_{n-k}=0) = 0$ for all $i \le n-k$, we obtain

$$
\begin{aligned}
&\left| G(j\omega) - \hat{G}_k(j\omega) \right| \\
&= \left| \sum_{i=1}^{n} c_i G_i(j\omega) - \sum_{i=1}^{n} c_i G_i(j\omega;b_{n-k}=0) \right| \\
&\le \sum_{i=1}^{n} |c_i| \{ G_i(0) - G_i(0;b_{n-k}=0) \} \\
&= \mathrm{abs}(\mathcal{C}) \left(-\mathcal{A}^{-1} \right) \mathcal{B} - \mathrm{abs}(\mathcal{C}_k) \left(-\mathcal{A}_k^{-1} \right) \mathcal{B}_k.
\end{aligned}
$$

Therefore, the result follows. The equality follows from the fact $\mathrm{abs}(\mathcal{C}) = \mathcal{C}$ (or $-\mathcal{C}$) if the entries of $\mathcal{C}$ have the same sign. ■

This theorem indicates that the upper bound of the approximation error in H_∞-norm can be obtained by only the matrix computation. In particular, if the all signs of the element of the matrix $\mathcal{C}$ are same, this gives the exact value.

Remark 1: We can easily determine $\mathcal{A}_k^{-1}$ in (18) by iterative calculation based on

$$
\mathcal{A}_{k+1}^{-1} = \begin{bmatrix} \mathcal{A}_k^{-1} + \zeta b_{n-k}^2 \Lambda\Lambda^{\mathsf{T}} & -b_{n-k}\zeta\Lambda \\ -b_{n-k}\zeta\Lambda^{\mathsf{T}} & \zeta \end{bmatrix}, \qquad (19)
$$

where $\zeta = 1/\{a_{n-k} - b_{n-k}^2 \mathcal{A}_k^{-1}(k,k)\}$ and $\Lambda = \mathcal{A}_k^{-1}(1:k,k)$.

Remark 2: The proposed method is mathematically similar to the model reduction based on Lanczos algorithm in Krylov method. However, Krylov method in general does not produce a tri-diagonal matrix in which off-diagonal terms have the *same sign*, since Lanczos algorithm is sensitive for rounding errors. In addition, it is known that if an system in question has uncontrollable states, the algorithm will stop due to occurring a division by zero. In fact, using the popular Krylov method we cannot transform the system matrix to a tri-diagonal matrix in numerical examples in sections IV-A and IV-B, since the reaction-diffusion systems on a network dealt with in this paper possibly have uncontrollable states due to the symmetry of their network. On the other hand, the proposed method can provide an order reduced model that can be regarded as a one-dimensional reaction-diffusion system, by using Householder transformation that is relatively robust for rounding errors to make all off-diagonal terms positive. Furthermore, by this positivity of off-diagonal terms the error bound via model reduction can be easily obtained.

The algorithm of the model reduction by the proposed method is as follows:

1) Transform the system matrices of the system (1) to **RD**-representation $(\mathcal{A},\mathcal{B},\mathcal{C})$ by applying Theorem 1
2) Give a positive constant ε as the upper bound of the approximation error
3) Find the minimum k satisfying

$$
\frac{\mathrm{abs}(\mathcal{C}_k) \mathcal{A}_k^{-1}\mathcal{B}_k - \mathrm{abs}(\mathcal{C}) \mathcal{A}^{-1}\mathcal{B}}{-\mathrm{abs}(\mathcal{C}) \mathcal{A}^{-1}\mathcal{B}} < \varepsilon, \qquad (20)
$$

   i.e., the estimation of the normalized approximation error
4) Construct $\hat{G}_k(s) = \mathcal{C}_k(sI_k - \mathcal{A}_k)^{-1}\mathcal{B}_k$

IV. Numerical examples

A. Model reduction for two-dimensional diffusion system

First, in order to understand the proposed method in an intuitive way, we consider a diffusion system on the square lattice composed of 20×20 nodes, as shown in Fig. 4. Suppose the system matrices (A, B, C) is given as follows: $A \in \mathbb{R}^{400\times 400}$ is given by

$$\begin{cases} \alpha_{ij} = 1 \ \ (\text{if node } i \text{ and } j,\ i \neq j,\ \text{are connected}) \\ \alpha_{ij} = 0 \ \ (\text{else},\ i \neq j) \end{cases}$$
$$\gamma_1 = 1, \quad \gamma_i = 0 \ \ \forall i,\ i \neq 1 \tag{21}$$

in (1), and $B \in \mathbb{R}^{400\times 1}$ and $C \in \mathbb{R}^{1\times 400}$ are given by

$$B = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{\mathsf{T}}, \tag{22}$$

$$C = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}, \tag{23}$$

i.e., the input is applied at the first node and the output is the sum of all states.

From (4), we obtain the **RD**-representation $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ for this system. Let us denote by $\mathcal{A}_{i,j}$ the i,j-th element of $\mathcal{A} \in \mathbb{R}^{400\times 400}$ and by c_i the i-th element of $\mathcal{C} \in \mathbb{R}^{1\times 400}$. Then in the upper figure of Fig. 5, for each k in the horizontal axis, the broken line shows the values of the reaction term $\sum_{i=1}^{400} \mathcal{A}_{k,i}$, i.e., the sum of the k-th row of $\mathcal{A}$, the line of $*$ shows the values of the diffusion term $\mathcal{A}_{k,k+1}$ $(= b_{400-k}$ in (3)), i.e., the off-diagonal entries of $\mathcal{A}$, and the solid line shows the values of $c_{400-k+1}$, i.e., the elements of $\mathcal{C}$. We can see from this figure that each element of $\mathcal{C}$ for around $k \geq 60$ has the value of almost "zero", and each element of the reaction term and the diffusion term decreases as k increases. This means that as the effect of the input on the state $x_{400-k+1}$ is weaker (i.e., as k increases), the self-dissolution of the corresponding state is stronger and b_{400-k} in (3), i.e, the intensity of the interaction between the states $x_{400-k+1}$ and x_{400-k}, is smaller. Therefore, it turns out that the states $x\,(k{+}1\!:\!400)$ have little relation to the input-output properties.

The lower figure of Fig. 5 expresses the values of the left-hand side of (20), i.e., the upper bound of the approximation error via model order reduction, for each k. This figure shows that the approximation error due to neglecting the states $x\,(54\!:\!400)$ is very small. In this example, we can apply (18) with equality in Theorem 4 since every element of $\mathcal{C}$ is positive. In fact, the minimum of k satisfying (20) is given by $k = 53$ when $\varepsilon = 0.05$. From Fig. 6, where the solid line denotes the Bode diagram of the original system (400-th order) and the symbols $*$ denote that of the approximated system at $k = 53$ (53-th order), we see that the both Bode diagrams are almost identical.

B. Model reduction for diffusion system on complex networks

Next, we consider a diffusion system on the complex network of a Barabasi-Albert model (exponent: 1.2) in Fig. 7, which is well-known as a graph satisfying the scale-free and small-world property [3]. This graph has 300 nodes and 300 edges in which some hubs are included and the first node is connected to every other node within 6 edges. Since in this graph all nodes are relatively close to the first node, at which the input is given, it is expected that the model order reduction method based on the proposed **RD**-representation is effective. The matrices $A \in \mathbb{R}^{300\times 300}$, $B \in \mathbb{R}^{300\times 1}$, and $C \in \mathbb{R}^{1\times 300}$ are given by (21), (22), and (23). Then in a similar way to the case of Fig. 5, the upper figure of Fig. 8 shows the reaction term and the diffusion term of $\mathcal{A} \in \mathbb{R}^{300\times 300}$ and the element of $\mathcal{C} \in \mathbb{R}^{1\times 300}$ in the **RD**-representation of (A, B, C). The lower one of Fig. 8 also shows the upper (actually, the exact) bound of the approximation error between the original system and the truncated system.

From Fig. 8, we find that each element of $\mathcal{C}$ have the value of almost "zero" for around $k \geq 40$, and the elements of $\mathcal{A}$ and $\mathcal{C}$ have relatively dispersed values for around $k \leq 150$ due to the complexity of the graph. This induces a negligible approximation error for around $k \geq 40$.

In fact, the minimum values of k satisfying (20) are given by $k = 33$ when $\varepsilon = 0.05$. In Fig. 9, the Bode diagram of the original system is denoted by the solid line, that of the truncated system by $*$. It turns out from this figure that the input-output property of the original system is appropriately approximated via the proposed method.

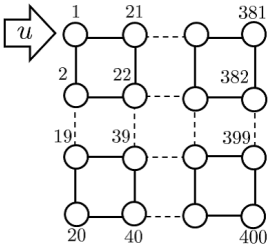

Fig. 4. Two dimensional diffusion (20×20)

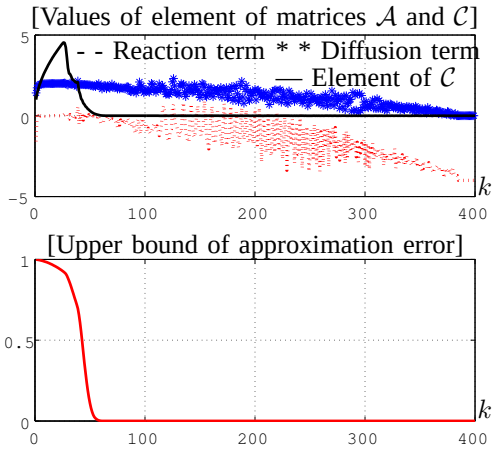

Fig. 5. Plots of values of matrices and upper bound of the approximation error

V. Conclusion

In this paper, we proposed a model order reduction method for SISO linear dynamical networks. In this method, the

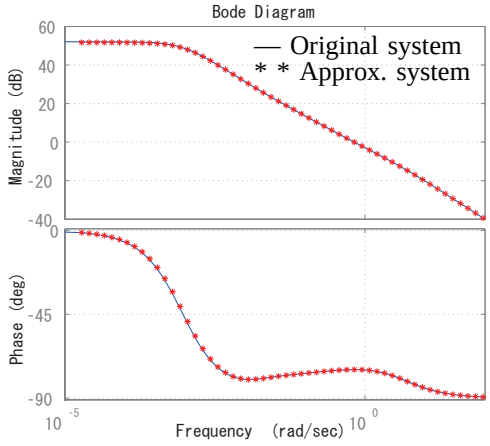

Fig. 6. Bode diagram

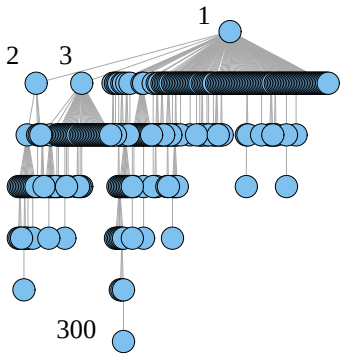

Fig. 7. Complex network of Barabasi-Albert model

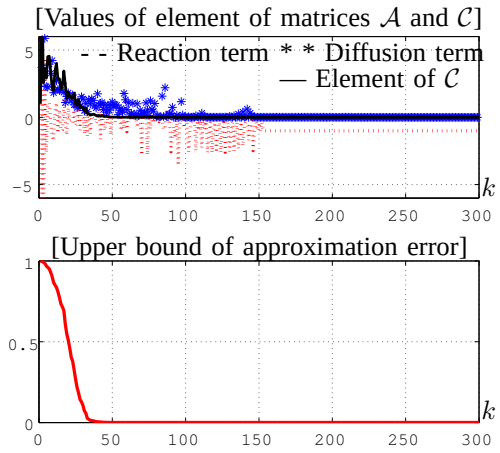

Fig. 8. Plots of values of matrices and upper bound of the approximation error

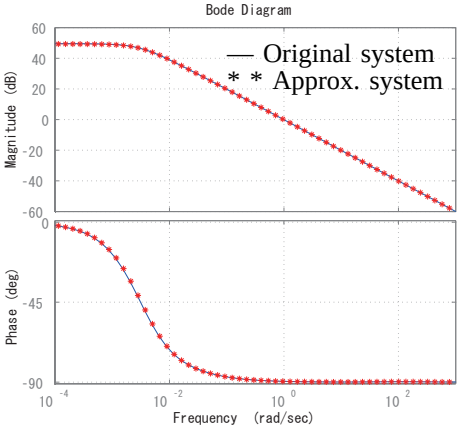

Fig. 9. Bode diagrams

structure of spatially one-dimensional reaction-diffusion that a SISO dynamical network has is extracted by way of a Householder transformation ordering the state variables according to the distance from the source (i.e., an input) of the diffusion. Based on this structure, a model order reduction method with the reaction-diffusion structure of the system preserved is presented. In addition, this model order reduction is systematically executed since the upper bound of the approximation error can be preliminarily obtained by simple matrix computations. The Householder transformation does not require computationally expensive operations, e.g., singular decompositions. In this sense, this method can be implemented for large-scale systems. An extension of this approach to the case of MIMO systems is one of future work.

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